Analytical Results for the Capacity of Spatially Correlated MIMO Channels

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Abstract—The influence of spatial correlation on the capacity of multiple-input, multiple-output (MIMO) channels is becoming increasingly important as terminals grow smaller and utilise more antenna elements. This has largely been studied through Monte-Carlo simulation methods. Where analytical methods have been considered, only bounds or approximations have been derived and only for the case of spatial correlation at one end of the radio link. In this paper exact analytical expressions are derived for the ergodic capacity of MIMO channels with spatial correlation at both ends and at both ends of the radio link. The latter case does not lend itself to numerical evaluation, but the former case is shown to be accurate by comparison with simulation results. The analysis is based on the transmitter and receiver antenna correlation matrices and can thus be applied to any spatially correlated MIMO channel.

I. INTRODUCTION
With the increasing demand for high data rate wireless communication services coupled with the problem of power and bandwidth restrictions as well as the harsh multipath fading channel, the field of high data rate, spectrally efficient and reliable wireless communications is currently receiving much attention.

It was shown in [1, 2] that multiple-input multiple-output (MIMO) channels, whereby multiple antennas are employed at both the transmitter and receiver, offer large gains in capacity over single-input single-output (SISO) channels. In order to exploit the benefits of MIMO channels, various coding schemes have been developed which include layered space-time codes [3], space-time trellis codes [4] and space-time block codes [5].

Multiple antenna systems are the most promising solution to achieving high data rates in future wireless communication systems. However spatial correlation between antenna elements degrades the performance of such systems. This effect becomes more pronounced as the terminal antenna spacing is reduced or more antenna elements are utilised. This is predicted to be a common problem in future wireless communication systems where there will be a high demand for small, high data rate terminals. Thus a good understanding of the effects of spatial correlation on the performance of such systems is essential.

The capacity of MIMO channels indicates the ultimate performance which can be achieved by multiple antenna systems using advanced coding and modulation techniques. It thus provides a good benchmark to assess the performance of such systems. MIMO capacity is generally investigated using Monte-Carlo simulation methods due to the difficulty experienced in deriving analytical expressions. However analytical expressions give a deeper understanding of the main factors affecting the channel capacity.

Where analytical expressions have been investigated, the ergodic capacity rather than the outage capacity was considered as the latter is mathematically more complicated. In [1] an exact analytical expression was derived for the ergodic capacity of an uncorrelated MIMO channel which was then evaluated using numerical integration. Many authors have proposed bounds or approximations as exact expressions are harder to derive. Approximations or bounds for the capacity of uncorrelated MIMO channels were investigated in [2, 6–9]. Spatial correlation between antenna elements reduces the capacity of MIMO channels. In [10–12] analytical bounds or approximations were derived for the case of spatial correlation at one end of the radio link.

To the extent of the authors knowledge no exact analytical expression has been derived for the case of spatial correlation at one end of the radio link and no analytical expression, bound or approximation has been derived for the case of spatial correlation at both ends of the radio link. In this paper analytical expressions are derived for the ergodic capacity of MIMO channels for the cases of spatial correlation at one end and at both ends of the radio link.

This paper is arranged as follows. In section II the system model is described. The channel model accounting for spatial correlation at both ends of the radio link is presented in section III. In section IV analytical expressions for the ergodic capacity of MIMO channels with correlation at one end and both ends of the radio link is derived. Section V assesses the accuracy of the analysis by comparing the analytical results to simulation results. Finally conclusions are drawn in section VI.

II. SYSTEM MODEL
Consider a narrowband, single user communications system with \( n_T \) transmit and \( n_R \) receive omnidirectional antenna elements. It is assumed the channel is known to the receiver and unknown to the transmitter which must equally divide the total transmit power over all antenna elements. The link between the transmit and receive antenna arrays is represented using the complex baseband vector notation

\[
y(t) = H(t)x(t) + z(t).
\]

The \( n_T \times 1 \) transmit vector \( x(t) \) has elements \( x_j(t) \) which denote the signal transmitted from antenna \( j =
\{1, \ldots, n_T\}\). The \(n_R \times 1\) receive vector \(y(t)\) has elements \(y_i(t)\) which denote the signal received on antenna \(i = 1, \ldots, n_R\). The \(n_R \times 1\) noise vector \(\mathbf{z}(t)\) has elements \(n_i(t)\) which denote the additive white Gaussian noise (AWGN) at receiver branch \(i\). The entries of the noise vector are independent and identically distributed complex Gaussian random variables with variance \(N_0\) where \(N_0\) is the noise power spectral density. Finally \(\mathbf{H}(t)\) is the \(n_R \times n_T\) channel matrix of complex path gains \(h_{ij}(t)\) between transmit antenna \(j\) and receive antenna \(i\).

### III. THE MIMO CHANNEL MODEL

In this section the spatially correlated MIMO channel model used in this paper is presented. This channel model was originally presented in [13] which modelled spatial as well as temporal correlation.

Consider the scenario where both the transmitter and receiver are surrounded by objects resulting in local scattering at both ends of the radio link and that no line-of-sight exists between the transmitter and receiver. Only local scattering is considered as it is assumed that the path loss will limit the contribution of remote scatterers to the total channel energy. It is assumed that the channel is memoryless and there is no correlation between consecutive symbols.

In Fig. 1 a geometric model for the MIMO channel under consideration is presented. Uniform linear arrays are employed at both the transmitter and receiver. The transmit antenna array lies at an angle \(\alpha\) from the x-axis with elements denoted by \(T_p\) spaced by \(\delta\). Similarly the receive antenna array lies at an angle \(\beta\) from the x-axis with elements \(R_m\) spaced by \(d\). Both the transmitter and receiver are surrounded by omnidirectional scatterers which lie on a ring of radius \(T\) and \(R\) respectively. \(TS(\theta)\) and \(RS(\phi)\) denote arbitrary transmit and receive scatterers at angles \(\theta\) and \(\phi\) from the x-axis.

Consider an arbitrary transmit antenna element \(T_p\) and an arbitrary receive antenna element \(R_m\). The plane waves transmitted from element \(T_p\) are scattered by the transmit scatterers and then by the receive scatterers which results in many rays impinging on element \(R_m\). One such ray is shown by the line joining \(T_{nr}\) to \(R_1\). Let \(h_{mp}(t)\) denote the path gain for the \(T_p - R_m\) link.

The correlation between two arbitrary links \(T_p - R_m\) and \(T_q - R_n\) is defined as follows

\[
\rho_{mp,nq}(t) = \rho_{mp,nq} = \frac{E[h_{mp}h_{nq}^*]}{\sqrt{\Omega_{mp}\Omega_{nq}}} \tag{2}
\]

where \(\Omega_{mp}\) and \(\Omega_{nq}\) is the power transferred through the links \(T_p - R_m\) and \(T_q - R_n\) respectively and \(\cdot^*\) is the complex conjugate. Note that the correlation is time independent as the channel is memoryless and for this reason the time index can be dropped.

The distribution of the scatterers is modelled using the von Mises pdf [14]. This is given below for the distribution of the transmit and receive scatterers \(p(\theta)\) and \(p(\phi)\) where \(\theta, \phi \in [-\pi, \pi]\)

\[
p(\theta) = \frac{1}{2\pi I_0(\kappa_T)} \exp[\kappa_T \cos \theta] \tag{3}
\]

\[
p(\phi) = \frac{1}{2\pi I_0(\kappa_R)} \exp[\kappa_R \cos(\phi - \pi)] . \tag{4}
\]

Here \(I_0(\cdot)\) is the modified Bessel function of the 1st kind of order zero and \(\kappa_T\) and \(\kappa_R\) control the spread of scatterers around the mean angle 0 and \(\pi\) respectively. When \(\kappa_T = 0\), \(p(\theta) = 1/2\pi\) is a uniform distribution and one has isotropic scattering. As \(\kappa_T\) increases, the scatterers become more clustered around the mean angle and the scattering becomes increasingly non-isotropic.

In [13] it was shown that the correlation between the two links can be expressed as the product of the transmit \(\rho_{pq}^T\) and receive \(\rho_{mn}^R\) antenna correlations as follows

\[
\rho_{mp,nq} = \rho_{pq}^T \cdot \rho_{mn}^R . \tag{5}
\]

Let \(b_{pq} = 2\pi(d(p-q))/\lambda\) and \(c_{mn} = 2\pi(d(m-n))/\lambda\) where \(\lambda\) is the carrier wavelength. By assuming \(T \gg \delta\) and \(R \gg d\), closed form expressions for the transmit and receive antenna correlations are given by

\[
\rho_{pq}^T \approx \frac{1}{I_0(\kappa_T)} \cdot I_0 \left\{ [\kappa_T^2 - b_{pq}^2]^{1/2} + 2\mu_T b_{pq} \cos \alpha \right\} . \tag{6}
\]

\[
\rho_{mn}^R \approx \frac{1}{I_0(\kappa_R)} \cdot I_0 \left\{ [\kappa_R^2 - c_{mn}^2]^{1/2} + 2\mu_R c_{mn} \cos(\beta - \pi) \right\} . \tag{7}
\]
IV. ANALYTICAL EXPRESSIONS FOR THE ERGODIC
CAPACITY

Both the outage and the ergodic capacity have been
used as capacity measures for the MIMO channel, how-
ever the latter is mathematically less complicated to
study analytically. The ergodic capacity measure is gen-
erally applied when the channel is memoryless, has no de-
lay constraints and the transmission time is long enough
to reveal the long term ergodic properties of the channel.

In this section analytical expressions are derived for
the ergodic capacity of a MIMO channel with spatial
correlation at one end and at both ends of the radio
link. This derivation is very general as it is based on the
transmit and receive antenna correlation matrices. The
simpler case of spatial correlation at one end of the radio
link is considered first and the more complicated case of
spatial correlation at both ends of the radio link is dealt
with next. Throughout this analysis it is assumed that
\( n_T \geq n_R \) although the analysis can be easily modified to
consider the case where \( n_T < n_R \) [1].

The ergodic capacity of the MIMO channel described
by the \( n_T \times n_T \) random channel matrix \( H \) of complex
path gains is calculated as follows [1]

\[
E[C] = E \left[ \log_2 \det \left( I_{n_T} + \frac{\rho}{n_T} H H^\dagger \right) \right]
\]

where \( \dagger \) is the transpose conjugate, \( \det(\cdot) \) is the matrix
determinant, \( I_{n_T} \) is the \( n_T \times n_T \) identity matrix and \( \rho \)
is the average SNR at each receive antenna. This can be
rewritten in terms of the eigenvalues \( \lambda_1, \ldots, \lambda_{n_R} \) of
\( A = HH^\dagger \) such that

\[
E[C] = E \left[ \sum_{i=1}^{n_R} \log_2 \left( 1 + \frac{\rho}{n_T} \lambda_i \right) \right].
\]

Let \( p(\lambda) \) denote the distribution of the eigenvalues of \( A \)
where \( \lambda = [\lambda_1, \ldots, \lambda_{n_R}] \). The ergodic capacity can be
calculated by integrating over \( p(\lambda) \) and is given by

\[
E[C] = \int_0^\infty \cdots \int_0^\infty \sum_{i=1}^{n_R} \log_2 \left( 1 + \frac{\rho}{n_T} \lambda_i \right) p(\lambda) \, d\lambda.
\]

Note that the limits of integration are 0 and \( \infty \) as the
eigenvalues of Hermitian matrix \( A \) are always non-
negative real numbers. By integrating over the distribu-
tion of the eigenvalues of \( A \) as opposed to the distribu-
tion of \( H \) the order of integration is reduced from \( 2n_T n_R \)
to \( n_R \).

When \( n_T \geq n_R \) the matrix \( A \) has a Wishart dis-
tribution with \( n_T \) degrees of freedom [15, 16]. However when
\( n_T < n_R \) \( A \) is singular and no distribution for \( A \)
exists. The transmit antenna correlation matrix is denoted
by \( \Sigma_T \) with elements \( \rho_{m,n}^T \) (6) and the receive antenna
correlation matrix is denoted by \( \Sigma_R \) with elements \( \rho_{m,n}^R \)
(7).

A. Spatial Correlation at the Receiver

The distribution of the eigenvalues of \( A \) is presented
here for the case of spatial correlation at the receiver.
This analysis can be easily modified to study spatial
correlation at the transmitter by interchanging the sub-
scripts \( T \) and \( R \). Here \( H \) is zero mean and normally dis-
tributed with independent columns and covariance matrix
\( \Sigma_R \).

In [16] the distribution of the ordered eigenvalues of
the complex Wishart distributed matrix was given. By
scaling this distribution by the factor \( 1/n_R \) the distribu-
tion of the unordered eigenvalues of the complex Wishart
distributed matrix \( A \) is given by

\[
p(\lambda) = C_1 \cdot \tilde{F}_0 \left( -\Sigma_R^{-1}, A \right) \cdot \prod_{i=1}^{n_R} \lambda_i^{n_T-n_R} \prod_{i<j} (\lambda_i - \lambda_j)^2
\]

\[
C_1 = \frac{\pi^{n_R(n_R-1)} \det(\Sigma_R)^{-n_T}}{n_R \Gamma(n_R(n_T)) \Gamma_{n_R}(n_R)}
\]

\[
\Gamma_m(n) = \frac{\pi^{m(m-1)/2}}{\prod_{i=1}^{m} \Gamma(n-i+1)}.
\]

Here \( \tilde{F}_0(\cdot, \cdot) \) is the hypergeometric function of two
matrix arguments, \( \Gamma_m(n) \) is the complex multivariate
gamma function, \( \Gamma(\cdot) \) is the gamma function and \( A = \text{diag}(\lambda_1, \ldots, \lambda_{n_R}) \). This expression would be straight-
forward to evaluate if it were not for the hypergeometric
function of matrix arguments which is generally ex-
pressed as a series expansion of zonal polynomials [15, 16].
However this formulation is not practical for numerical
work as zonal polynomials are well known for being ex-
tremely difficult to compute and the series expansion is
slow to converge. A more tractable formulation was pro-
posed in [17] where the hypergeometric function of two
matrix arguments was expressed in terms of classical hy-
pergeometric functions as follows

\[
\tilde{F}_0(S, T) = \frac{\det(\tilde{F}_0(s_i t_j)) \Pi_{i=1}^{n}(j-1)!}{V(S) V(T)}
\]

\[
\det(\tilde{F}_0(s_i t_j)) = \det \left( \begin{array}{cccc}
\tilde{F}_0(s_1 t_1) & \cdots & \tilde{F}_0(s_1 t_n) \\
\tilde{F}_0(s_2 t_1) & \cdots & \tilde{F}_0(s_2 t_n) \\
\vdots & \ddots & \vdots \\
\tilde{F}_0(s_n t_1) & \cdots & \tilde{F}_0(s_n t_n)
\end{array} \right)
\]

\[
V(S) = \prod_{i<j}(s_i - s_j)
\]

\[
V(T) = \prod_{i<j}(t_i - t_j)
\]

where \( S \) and \( T \) are \( n \times n \) Hermitian matrices with real
eigenvalues \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_n \) respectively. The
function \( \tilde{F}_0(x) = \exp(x) \) is the classical hypergeometric
function. This formulation requires that the eigenvalues
of the matrices \( S \) and \( T \) are unequal but this does not
pose a problem as discussed later. By substituting (12)
to (11), the distribution of the unordered eigenvalues
of \( A \) can be written as

\[
p(\lambda) = C_2 \cdot \exp(\lambda_i \sigma_j) \cdot \prod_{i=1}^{n_R} \lambda_i^{n_T-n_R} \prod_{i<j} (\lambda_i - \lambda_j) \cdot \frac{\pi^{n_R(n_R-1)} \det(\Sigma_R)^{-n_T}}{n_R \Gamma(n_R(n_T)) \Gamma_{n_R}(n_R)} \cdot \frac{\Pi_{i=1}^{n_R}(j-1)!}{V(-\Sigma_R^{-1})}
\]

\[
C_2 = \frac{\pi^{n_R(n_R-1)} \det(\Sigma_R)^{-n_T}}{n_R \Gamma(n_R(n_T)) \Gamma_{n_R}(n_R)} \cdot \frac{\Pi_{i=1}^{n_R}(j-1)!}{V(-\Sigma_R^{-1})}
\]

\[
\]
where the $\sigma_j$’s are the eigenvalues of $-\Sigma_R^{-1}$. Note that the problem of equal eigenvalues of $\Lambda$ is avoided because $V(A)$ is cancelled out by the 2nd product term of (11). Furthermore the eigenvalues of the correlation matrix $\Sigma_R$ as well as its inverse are unequal for most practical cases.

The ergodic capacity can now be calculated by substituting $p(\lambda)$ in (10) with (13) and evaluating the integral numerically.

**B. Spatial Correlation at the Transmitter and Receiver**

In this section the distribution of the eigenvalues of $A$ is presented where spatial correlation at the transmitter and receiver is accounted for. In this case $H$ is zero mean and normally distributed where $\Sigma_T$ and $\Sigma_R$ are the covariance matrices of the rows and columns respectively.

The distribution of $A$ is a special case of the quadratic form of normal vectors and has been studied by a number of authors [18, 19]. In [19] the distribution of the eigenvalues of the quadratic form of complex normal vectors was derived. The distribution of the unordered eigenvalues of authors [18,19]. In [19] the distribution of the eigenvalues of $\Lambda$ is presented where spatial correlation at the transmitter and receiver is accounted for. In this case $H$ is zero mean and normally distributed where $\Sigma_T$ and $\Sigma_R$ are the covariance matrices of the rows and columns respectively.

The distribution of $A$ is a special case of the quadratic form of normal vectors and has been studied by a number of authors [18,19]. In [19] the distribution of the eigenvalues of the quadratic form of complex normal vectors was derived. The distribution of the unordered eigenvalues of $A$ is obtained by setting the constant matrices in [19, Sec. 8] to identity matrices and scaling the distribution by $1/n_R!$. Hence $p(\lambda)$ is given by

$$p(\lambda) = C_3 \cdot \tilde{F}_0(\Sigma_T^{-1}, \Lambda, -\Sigma_R^{-1}) \prod_{i=1}^{n_R} (\lambda_i - \lambda_j)^2$$

$$C_3 = \frac{\pi^{n(n-1)} \det(\Sigma_T)^{-n_T} \det(\Sigma_R)^{-n_R}}{n_R! \Gamma_{n_T} \Gamma_{n_R}}$$

where $\tilde{F}_0(\cdots)$ is the hypergeometric function of three matrix arguments. This function can be evaluated using a series expansion of zonal polynomials but as discussed previously, this formulation is not practical for numerical work. As far as the author is aware no practical formulation for this function exists which can be evaluated numerically with realisable complexity.

**V. RESULTS**

In this section the analytical expression derived in section IV for the ergodic capacity of a MIMO channel with spatial correlation at the receiver is compared to simulation results. Furthermore the method followed to generate spatially correlated Rayleigh random variables is described.

**A. Generation of the Correlated Variates**

The following method is used to generate spatially correlated variates with correlation $\Sigma_R$. For each time instance a $n_R \times n_T$ matrix $U$ of independent zero mean complex Guassian random variables with unit variance is generated. The channel matrix $H$ with the desired spatial correlation is generated as follows

$$H = \sqrt{\Sigma_R} U.$$  

(15)

The matrix square root of $\Sigma_R$ is calculated from the eigenvalues $\sigma_i$ and eigenvectors $\xi_i$ of $\Sigma_R$ as shown below

$$\sqrt{\Sigma_R} = X \begin{bmatrix} \sqrt{\sigma_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\sigma_{n_R}} \end{bmatrix} X^\dagger$$

(16)

**B. Analytical Results**

The accuracy of the analytical results of section IV is assessed here by comparison with simulation results for the case of spatial correlation at the receiver only. In all cases the ergodic capacity $E[C]$ is used as the capacity measure. The simulation results are generated using Monte-Carlo methods where many instances of the channel $H$ are generated according to (15) and the ergodic capacity is calculated using (8). The analytical expression (10) with (13) substituted for $p(\lambda)$ is evaluated numerically using a multi-dimensional adaptive step size integration technique with limits of integration of 0 and 100 for each dimension.

Let $n$ describe the case of equal transmit and receive antennas i.e. $n = n_T = n_R$. Unless indicated otherwise, the following parameters are used. The SNR is set to 10dB. The receiver antenna spacing is $d = \lambda$ and the antenna array angle is $\beta = 45^\circ$. The degree of local scattering at the receiver is $\kappa_R = 10$. The carrier frequency is set to 1GHz which corresponds to a carrier wavelength $\lambda$ of 0.3m.

In Fig. 2 and Fig. 3 the distribution of the eigenvalues $p(\lambda_1, \lambda_2)$ of $A$ is given for a $n = 2$ system for $d = \lambda$ and $d = 0.1\lambda$ respectively. When $d = \lambda$ the two peaks are small and spread out while for a smaller spacing of $d = 0.1\lambda$ the peaks are higher and sharper due to the increase in spatial correlation. These sharp peaks increase the computational complexity of the numerical integration as a smaller step size is required to obtain accurate results. Thus for large dimension systems with high spatial correlation, it is not computationally feasible to obtain analytical results using this analysis.

![Fig. 2. Distribution of the eigenvalues of $A$ for $d = \lambda$](image-url)

In figures 4-6 the analytical results are compared to simulation results for the $n = 2$ and $n = 3$ systems. In all cases the analysis agrees well with the simulation results. In Fig. 4 it is observed that a reduction in antenna spacing results in a decrease in capacity as there is a higher degree of correlation between antenna elements. Increasing the antenna spacing beyond $2\lambda$ has a negligible effect on capacity for both the $n = 2$ and $n = 3$ systems.
Fig. 3. Distribution of the eigenvalues of A for $d = 0.1\lambda$.

The influence of the receiver array angle is investigated in Fig. 5. It is noted that the capacity is at a minimum when the array is parallel to the wavefront arrival ($\beta = 0^\circ$) and at a maximum when the array is perpendicular to the wavefront arrival ($\beta = 90^\circ$). For ($\beta > 60^\circ$) the corresponding increase in capacity is negligible.

Fig. 4. Analytical results for the ergodic capacity of a n transmit, n receive antenna system with varying antenna spacings $d/\lambda = 0.1, 0.2, 0.4, 0.6, 0.8, 1, 2, 3, 4, 5$.

VI. Conclusions

In this paper analytical expressions were derived for the ergodic capacity of MIMO channels with spatial correlation at one end and at both ends of the radio link. For the case of correlation at one end of the radio link, the analytical expression was evaluated using a multidimensional numerical integration technique. The analysis agreed well with simulation results for a range of channel parameters for both the $n = 2$ and $n = 3$ systems. For the case of correlation at both ends of the radio link, the analytical expression derived contained a hypergeometric function of 3 matrix arguments which could not be evaluated numerically with realisable complexity. This analysis is based on the transmitter and receiver antenna correlation matrices and can thus be applied to any spatially correlated MIMO channel.

Future work may consider the approximation of the hypergeometric function of 2 matrix arguments to allow a closed form expression for the ergodic capacity to be derived. Furthermore the calculation or approximation of the hypergeometric function of 3 matrix arguments would be a useful.

REFERENCES


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