Lagrangian Relaxation as a solution approach to solving the Survivable Multi-Hour Network Design Problem

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***Abstract*— Survivable multi-hour network design aims at finding a cost efficient network design that is robust enough to operate under varying traffic requirements as well as in the case of single link failure. Based on a mixed-integer programming formulation, we present a simple heuristic which employs dual information from a Lagrangian relaxation. Computational results are compared to a Branch-and-Cut approach for experimental data.**

I. Introduction

In addition to the objective of designing cost efficient telecommunication networks, robustness and survivability have become major considerations for telecommunication operators in satisfying quality of service (QoS) requirements.

A robust telecommunication network should be able to support noncoincident peak traffic even in the event of component failures. The network design problem therefore has to take into account multiple non-simultaneous traffic demand matrices by simultaneously satisfying survivability requirements.

Several papers have been published that address the problem of network dimensioning with multi-hour traffic conditions, but the underlying model assumptions and application areas distinguish them. Some of the earlier models for multi-hour network design have simplifying assumption such as continuous capacities (see e.g. [1], [2] and [3]). These type of models are computationally very attractive but are not practical since communication equipment cannot be installed in fractional quantities. It is for this reason then that we are only interested in models having integrality restrictions on at least the capacity variables.

Several solution approaches for solving the multi-hour network design problem with integrality restrictions have been proposed. A Lagrangian decomposition approach with subgradient optimization has been suggested by e.g. [4], [5], and [6]. In [7] a block surrogate relaxation of the problem is presented and heuristics are provided for finding primal solutions. In [8] a Benders decomposition followed by a so-called 3-phase heuristic is applied and in both [9] and [10] the problem is formulated as a two stage stochastic programming problem with recourse. In [9] an analytic center cutting plane method is proposed and in [10] a Benders decomposition approach. In [11] a Branch-and-cut approach is followed that incorporates several classes of valid inequalities.

Survivability with respect to network design can be modeled in various ways [12],[13]. The basic principle however remains to design a network that is able to partially or completely satisfy demand requirements in the event of component failures. In [14] the multi-hour problem for the design of IP networks is presented. To the best of our knowledge this is the only paper addressing multi-hour problems that takes into account survivability requirements. The scope of the paper, however, is restricted to shortest path based routing protocols such as OSPF.

The contribution of this paper is towards exploring the use of a Lagrangian relaxation approach for solving the survivable multi-hour network design problem. Results from the Lagrangian relaxation approach are compared to results obtained from solving the problem with a branch-and-cut scheme that incorporates several classes of valid inequalities.

The content of this paper is organized as follows: In Section II an overview of the model is presented which comprises two sub-problems, the hardware configuration and the routing problem. In Section III details of the algorithmic approach are presented and attention is drawn to a Lagrangian relaxation framework. Some implementation details are presented in Section IV and computational results are presented in Section V based on randomly generated data. Finally a summary and some concluding remarks are given in Section VI.

II. The survivable multi-hour network design model

The survivable multi-hour network design problem is modeled as a mixed-integer programming problem that comprises two parts, the hardware model and the routing model.

A. The hardware model

The hardware model allows a detailed representation of potential hardware configurations found in typical SDH or opaque WDM networks [15]. Potential technologies are abstracted into so called node designs and the optimization process is responsible for selecting the most cost effective node design for each node that will satisfy capacity requirements. Similarly, link designs enable the modeling of different transmission technologies that may have different capacities, cost structures and port requirements.

Let $G = (V, E)$ be the supply graph where V is the set of potential node locations and E is the set of potential communication edges. At each node $v \in V$ a set of admissible node designs $\mathcal{N}(v)$ are defined, and at most one node design $n \in \mathcal{N}(v)$ can be selected for installation. Likewise, for each edge $e \in E$ at most one link design $l \in \mathcal{L}(e)$ can be selected for installation from the predefined set of admissible link designs $\mathcal{L}(e)$.

The attributes of a node design $n \in \mathcal{N}(v)$ are, a set of installable module types $\mathcal{M}(n)$, a limit $M^{m,n} \in \mathbb{Z}_+$ on the number of modules of type $m \in \mathcal{M}(n)$ that may be installed, the available number of slots $S^n \in \mathbb{Z}_+$ that can be occupied by one or more modules, a switching capacity $C^n \in \mathbb{Z}_+$, and a cost $c^n \in \mathbb{R}_+$. Furthermore, each type of module $m \in \mathcal{M}(n)$ is designed to occupy a number of slots $S^m \in \mathbb{Z}_+$ and can accommodate several types of interfaces $\mathcal{I}(m)$. Each module also has a cost of $c^m \in \mathbb{R}_+$.

To enable the matching of link technologies with node technologies, each link design $l \in \mathcal{L}(e)$ for an edge $e \in E$ has a set of interface types $\mathcal{I}(l)$. Matching the interface types of a module m on a node n with the interface types of an edge e allows us to connect the edge e with the node n. There is a limit $I_i^m \in \mathbb{Z}_+$ on the number of interfaces of type $i \in \mathcal{I}(m)$ for a module m and a limit $I_i^l \in \mathbb{Z}_+$ on the number of interfaces of type $i \in \mathcal{I}(l)$ for a link design l.

For ease of notation let $\mathcal{M}(v) := \bigcup_{n \in \mathcal{N}(v)} \mathcal{M}(n, v)$ denote the set of modules installable at node $v \in V$ and $\mathcal{I}(v) := \bigcup_{m \in \mathcal{M}(v)} \mathcal{I}(m)$ the set of potential interfaces at node $v \in V$.

The following decision variables are required to model the installation of the node designs, the modules, and the link designs.

- $x_v^n \in \{0,1\}$ indicates whether node design $n \in \mathcal{N}$ is installed at node $v \in V$ or not.
- $x_e^l \in \{0,1\}$ indicates whether link design $l \in \mathcal{L}$ is installed on edge $e \in E$ or not.
- $x_v^m \geq 0$ integer, the number of modules of type $m \in \mathcal{M}$ installed at node $v \in V$.

Subsequently, the objective function of the hardware model is to minimize the installation cost of the node designs, the modules, and the link designs.

$$\min \sum_{v \in V} \left(\sum_{n \in \mathcal{N}(v)} c^n x_v^n + \sum_{m \in \mathcal{M}(v)} c^m x_v^m \right) + \sum_{e \in E} \sum_{l \in \mathcal{L}(e)} c^l x_e^l$$

The following set of constraints defines the rules for installing the necessary hardware that will satisfy a capacity vector y_e.

$$\mathcal{H} = \Bigg\{ x \in \mathbb{Z}_+^{|\mathcal{N}|\times|V|+|\mathcal{L}|\times|E|+|\mathcal{M}|\times|V|} :$$

$$\sum_{n \in \mathcal{N}(v)} x_v^n \leq 1 \quad \forall v \in V \tag{1}$$

$$\sum_{l \in \mathcal{L}(e)} x_e^l \leq 1 \quad \forall e \in E \tag{2}$$

$$\sum_{l \in \mathcal{L}(e)} C^l x_e^l = y_e \ \forall e \in E \tag{3}$$

$$\sum_{e \in \delta(v)} \sum_{l \in \mathcal{L}(e)} C^l x_e^l - \sum_{n \in \mathcal{N}(v)} C^n x_v^n \leq 0 \quad \forall v \in V \tag{4}$$

$$\sum_{m \in \mathcal{M}(v)} S^m x_v^m - \sum_{n \in \mathcal{N}(v)} S^n x_v^n \leq 0 \quad \forall v \in V \tag{5}$$

$$\sum_{e \in \delta(v)} \sum_{l \in \mathcal{L}(e)} I_i^l x_e^l - \sum_{m \in \mathcal{M}(v)} \sum_{i \in \mathcal{I}(m)} I_i^m x_v^m \leq 0 \quad \begin{matrix} \forall v \in V, \\ \forall i \in \mathcal{I}(v) \end{matrix} \tag{6}$$

$$x_v^m - \sum_{n \in \mathcal{N}(v)} M^{m,n} x_v^n \leq 0 \quad \begin{matrix} \forall v \in V, \\ \forall m \in \mathcal{M}(v) \end{matrix} \Bigg\} \tag{7}$$

Constraints (1) and (2) enforces the rule that only one node design be assigned to a node and only one edge design be assigned to an edge respectively. Constraint sets (3) and (4) ensure that the capacity of a node is sufficient to switch all the capacity of its incident edges. Note that y_e is an auxiliary variable that gives the edge capacity resulting from a valid inequality generated from the solution of the routing model (see subsequent section). To ensure that the slot requirements of installed modules do not exceed the available slots of a node design, constraint set (5) is applied. Constraint set (6) will ensure that the number of interfaces of type $i \in \mathcal{I}(v)$ that will be installed at node v is sufficient to accommodate the number of interfaces of type $i \in \mathcal{I}(l)$ required by the incident edges. An upper bound on the maximum number of modules of type m at a node v is defined by the constraint set (7).

B. The routing model

The objective of the routing model is to provide the hardware model with the capacity requirements on the communication links by finding feasible routings that will satisfy restrictions like hop count limits, survivability requirements etc. The overall model is, therefore, responsible for selecting an optimal hardware configuration and, simultaneously, finding feasible routings such that all point-to-point traffic requirements are satisfied.

In order to model the demand requirements between a pair of nodes the notion of a commodity is introduced. That is, the undirected demand between a node pair corresponding to a commodity $k \in \mathcal{K}$ is denoted as d_k. The set $\mathcal{T} = \{1, 2, \ldots, T\}$ is used to index the set of demand vectors $D = \{d^1, d^2, \ldots, d^T\}$.

The concept of failure states is used to facilitate the modeling of survivability. Different failure scenarios can be modelled by letting each failure state $s \in S$ contain all the network components (nodes and edges) that fail simultaneously. Survivability could be achieved by considering two alternatives [16]. The first is to introduce a *diversification* parameter $\delta_k \in (0, 1] \in \mathbb{R}_+$ that defines a limit on the fraction of the demand d_k that can be routed through any node or edge that is affected by a failure state. The result is that traffic is distributed more evenly across the network and in the case of component failure, traffic can be redirected through surviving paths. The second alternative is to impose a path length restriction. The parameter $\gamma_k \in \mathbb{N}_+$ is an upper bound on the number of edges being traversed by any path that routes (part of) the demand d_k. The idea is to avoid long paths that have a higher probability of being affected in the case of component failures. In this paper we employed the diversification parameter model and considered only single edge failure states.

Let $p \in \mathcal{P}$ be a set of edges that defines a non-cyclic undirected path between a communicating node pair, with $\mathcal{P}$ containing all possible paths for all possible node pairs. The subset $\mathcal{P}(k) \subseteq \mathcal{P}$ contains all paths that can route traffic between the communicating node pair corresponding to the commodity $k \in \mathcal{K}$. The set $\mathcal{P}(k, s) \subseteq \mathcal{P}(k)$ is used to index all paths for a commodity k that are effected by a failure state $s \in S$, and the set $\mathcal{P}(k, e) \subseteq \mathcal{P}(k)$ is required to index all paths for a commodity k that traverse an edge $e \in E$. The variables $r_p \in \mathbb{R}_+$ are introduced to define the fraction of flow for traffic on path $p \in \mathcal{P}$.

For a given capacity vector $y \in \mathbb{R}_+^{|E|}$ and a set of demand vectors $D = \{d^1, d^2, \ldots, d^T\}$ the following polyhedron $P(y, D, \delta)$ defines all feasible routings of the survivable multi-hour network flow problem:

$$P(y, D, \delta) = \Bigg\{ r \in \mathbb{R}_+^{|\mathcal{P}| \times T} :$$

$$\sum_{p \in \mathcal{P}(k)} r_p^t = 1, \ \forall\, k \in \mathcal{K}, \ \forall\, t \in \mathcal{T} \tag{8}$$

$$\sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}(k,e)} r_p^t d_k^t \le y_e, \ \begin{matrix} \forall\, e \in E, \\ \forall\, t \in \mathcal{T} \end{matrix} \tag{9}$$

$$\left. \sum_{p \in \mathcal{P}(k,s)} r_p^t d_k^t \le \delta_k d_k^t, \ \begin{matrix} \forall\, k \in \mathcal{K}, \\ \forall\, s \in \mathcal{S}, \\ \forall\, t \in \mathcal{T} \end{matrix} \right\} \tag{10}$$

If $P(y, D, \delta) \neq \emptyset$ a set of feasible routings exists such that the demand vectors $d^1, d^2, \ldots, d^T$ can be routed non-simultaneously. In this case the capacity vector y *supports* the set of demand vectors D with a common diversification vector δ.

The preceding definition of $P(y, D, \delta)$ implies a *dynamic routing* model since the demand vectors $d^1, d^2, \ldots, d^T$ can be routed independently of each other according to the routings $r^1, r^2, \ldots, r^T$. In some cases, due to different planning requirements, *static routing* might be preferred where the same routing is applied to all demand vectors. To enforce static routing for the survivable multi-hour network flow problem, the index t for the variable r can be discarded in the definition of $P(y, D, \delta)$ resulting in only one set of routing variables defining the same routing for all of the demand vectors.

III. Solution approach

The contribution of this paper is towards exploring the use of a Lagrangian relaxation approach for solving the survivable multi-hour network design problem. For purposes of comparison the survivable multi-hour network design problem is also solved with a branch-and-cut scheme that incorporates several classes of valid inequalities. The separation procedure employed in the branch-and-cut scheme for generating metric inequalities entails solving linear programs at each branch-and-bound node in order to determine if $P(y, D, \delta) \neq \emptyset$, with y the current solution to the hardware model and D the set of multi-hour demand vectors.

The routing polyhedron $\mathcal{F} \supseteq P(y, D, \delta)$ is obtained by removing the capacity constraints (9) from $P(y, D, \delta)$. By adding the relaxed capacity constraints to the objective function of the overall hardware and routing model we obtain the following Lagrangian relaxation formulation:

$$\begin{aligned} (\mathrm{LR}): \ \min\ & cx + \sum_{t \in \mathcal{T}} \sum_{e \in E} \mu_e^t \left(\sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}(k,e)} r_p^t d_k^t - y_e \right) \\ & \mu_e^t \ge 0, \ \forall\, t \in \mathcal{T}, \ \forall e \in E \\ & x \in \mathcal{H} \\ & r \in \mathcal{F} \end{aligned}$$

where c is the cost vector associated with the hardware components x and μ the lagrangian multipliers associated with the capacity constraints (9).

By re-arranging the terms of LR two separable functions $\Phi_1(\mu)$ and $\Phi_2(\mu)$ can be defined in terms of x and r respectively. That is,

$$\Phi_1(\mu) = \min_{x \in \mathcal{H}} \left\{ cx - \sum_{e \in E} \left(\sum_{t \in \mathcal{T}} \mu_e^t \right) y_e \right\}$$

$$\Phi_2(\mu) = \min_{r \in \mathcal{F}} \left\{ \sum_{t \in \mathcal{T}} \left(\sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}(k)} \left(\sum_{e \in E(p)} \mu_e^t \right) r_p^t d_k^t \right) \right\}$$

The Lagrangian relaxation problem LR is solved by applying a subgradient optimization technique. In each iteration of the subgradient optimization technique the gradient involved is updated by the solutions from the two functions $\Phi_1(\mu)$ and $\Phi_2(\mu)$. It is therefore advantageous if the two functions can be solved as fast as possible. Solving $\Phi_1(\mu)$ is unfortunately computationally expensive since it requires solving a mixed integer programming problem. For this reason is it being approximated by solving only the linear programming relaxation. Furthermore, since solving $\Phi_2(\mu)$ to optimality would require a column generation approach to manage the exponential number of path variables, do we approximate $\Phi_2(\mu)$ by solving an algorithm [17] that produces $\lceil 1/\delta_k \rceil$ number of disjoint shortest paths for each $k \in \mathcal{K}$.

The solution to the Lagrangian relaxation only yields a lower bound. A heuristic is therefore applied to generate primal solutions during the course of optimization. As a first attempt a very simple heuristic is considered. Subsequent to each iteration of the subgradient optimization a solution to the edge capacities, say y^*, is obtained by solving $\Phi_1(\mu)$. These edge capacities do not necessarily satisfy the original capacity constraints, but it is expected that for the final solution of LR, the edge capacities y^* would be close to the optimal solution. The heuristic therefore entails solving the hardware model with $y \geq y^*$, provided that $P(y^*, D, \delta) \neq \emptyset$. A solution found for the hardware model with lower bounds y^*, would constitute a feasible solution to the overall survivable multi-hour network design problem.

IV. Implementation Details

The implementation of the Lagrangian approach was done in C++ and is based on the network design tool DISCNET (see [13] and [18]). A Conic Bundle solver [19] was used for solving the subgradient optimization problem. For purposes of comparison, results for the branch-and-cut approach were generated by using the commercial mathematical programming software CPLEX.

As indicated in the preceding section, the dual function $\Phi_1(\mu)$ is approximated by solving only the linear programming relaxation. From computational experiences, however, it was found that by solving the linear programming relaxation with additional separated valid inequalities improved the situation. This was achieved by solving $\Phi_1(\mu)$ with a Cutting Plane Algorithm [20].

V. Computational Results

The potential supply graph considered for the empirical tests has 32 nodes and 64 potential links. The multiple demand vectors were created randomly from a uniform distribution. Data sets consisting out of 100, 200, 400 and 800 demand vectors were generated.

In terms of the hardware model, only one potential node design was specified for each of the nodes in the network. This node design can host up to five different modules, with each module compatible to a different interface. To enable modeling different edge capacities a total of 18 different link designs were considered. Each link design has in addition to different cost structures different interface requirements.

Vectors	First Sol.	Tot-Time	%Gap	Lower B.	Upper B.
100	0:16:37	3:02:11	11.5	227796153	254086807
200	0:33:45	3:02:06	13.8	226666746	257940889
400	1:34:13	2:59:10	13.1	224696581	254224273
800	-	3:03:02	-	224327079	-

TABLE I
Computational Results - CPLEX

Vectors	First Sol.	Tot-Time	%Gap	Lower B.	Upper B.
100	0:00:46	0:03:43	30.3	223876236	291595438
200	0:01:20	0:09:06	34.7	224453257	302432913
400	0:05:19	0:18:33	31.2	224720889	294910759
800	0:13:27	1:00:18	30.2	228976362	298118450

TABLE II
Computational Results - Lagrangian Relaxation

For modeling survivability the diversification parameters δ_k, for all $k \in \mathcal{K}$ were set to 0.5.

The computational results for the branch-and-cut approach and the Lagrangian relaxation approach are displayed in Table I and Table II respectively. The column labeled *First Sol.* in both the tables indicate the time elapsed until the first feasible solution was found. The column labeled *Tot-Time* indicates the total running time for the respective data sets, and the columns *Lower B.* and *Upper B.* shows the lower and upper bounds respectively. The column *%Gap* is an indication of the quality of the best solution found and is calculated as $\frac{\text{UpperB.}-\text{LowerB.}}{\text{LowerB.}} \times 100$. All experimental runs were limited to a total running time of 3 hours.

As a first observation notice that no feasible solution could be found for the data set of 800 demand vectors within the time limit of 3 hours for the branch-and-cut approach. Furthermore, it is clear from Table II that the lagrangian relaxation approach is superior in finding an initial solution in the shortest possible time. In fact, the lagrangian relaxation approach terminated for all data instances even before the branch-and-cut approach could find an initial solution.

The downside, however, is that for all data instances the *%Gap* for the lagrangian relaxation approach was larger compared to that of the branch-and-cut approach.

VI. Summary and Conclusion

In this paper we considered the survivable multi-hour network design problem where variations in network traffic are being represented by a collection of non-simultaneous demand vectors. The hardware part of the model encapsulates detailed technology requirements, and diversification restrictions are being imposed on the routing to satisfy survivability requirements.

A simple heuristic has been suggested within a Lagrangian relaxation framework and computational results are based on randomly generated data. For purposes of comparison results were also computed using the commercial mathematical software package CPLEX.

From the results it is clear that the proposed heuristic provides solutions in a very short period of time. For all data instances, the heuristic produced final solutions much faster compared to CPLEX, although not of such good quality.

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