Abstract—This paper compares two methods of parameter estimation for lognormal raindrop size distribution model in Durban, South Africa. The methods are the maximum likelihood estimation (MLE) and the method of moments (MoM). Parameter estimates with the method of moments is found to give best fit compared with the maximum likelihood estimation technique. Large deviation error was observed between the measured and modelled raindrop size distributions with the MLE estimates over the rainfall rates considered. A comparison of the proposed lognormal model was made with different tropical regions and the results compare favourably well with these other locations.

Index Terms—raindrop size distribution, maximum likelihood estimation, method of moments and root mean square error.

I. INTRODUCTION

The ever increasing rate of network failure due to poor service caused by attenuation and scattering of radio waves by raindrops has generated misunderstandings between service providers and the consumers. Often times, this attenuation becomes more serious and severe at microwave and millimetric wave spectrum (30-300GHz). These disturbances have led researchers to the study of these phenomena, leading to modeling of the raindrop size distributions (RDSD) for the temperate and tropical regions. According to [1], the raindrop size distribution model is defined as the probability density of equivolumetric drop diameter, \( D \) being in the unity volume. It is the distribution relating to raindrop sizes and mostly expressed in an exponential form.

Notable models have been developed and proposed for the RDSD models in different regions across the globe. The lognormal distribution model was found to be suitable for modeling tropical raindrop size distributions, and adequate for calculating the specific attenuation in the tropical regions whereas the negative exponential model of Marshall and Palmer (M-P) was adopted as best fit for RDSD in the temperate regions [2-4]. In this paper, we analyze the lognormal DSD model in Durban (29\(^\circ\)52\('\)S, 30\(^\circ\)58.9\('\)E), South Africa. The raindrop size is an essential micro-structural property in the modeling and prediction of rain attenuation. The statistical behavior of raindrops can therefore serve as a major criterion for choosing an appropriate distribution or, indeed the normal distribution [5]. The description of RDSD in the tropical regions is being hindered by a lack of adequate data that will cover the prominent regional and seasonal variability of average air motion and turbulence responsible for the development of precipitation and subsequent coalescence with breakup of raindrops [6].

A well-known integral equation exits between RDSD or number density \( N(D) \), terminal velocity \( V(D) \), and the drop diameter, \( D \), with rainfall rate \( R \). This relation can be expressed by equation (1) as [7]:

\[
R = 6\pi \times 10^{-4} \int_0^\infty D^3 V(D) N(D) \Delta D \quad [\text{mm/h}]
\]

where \( D \) is the rainfall diameter. \( N(D) \) is the number of drops per unit volume with diameter intervals between \( D \) and \( D+dD \) and \( V(D) \) is the terminal velocity of rain drop in \( \text{m/s} \) derived from the Gunn and Kinzer’s [7] terminal velocity of water drop. Equation (1) must not be violated by any RDSD.

It is important to note that the dropsizes distributions play a significant role in the estimation of the specific attenuation due to scattering and absorption of raindrops.

Generally, the specific attenuation \( A_s \) (in \( \text{dB/km} \)) is given by the relation [6]:

\[
A_s = 4.343 \times 10^{-3} \int_0^\infty Q_e N(D) dD \quad [\text{dB/km}]
\]

where \( Q_e \) is the total extinction cross-section, which is the sum of the power scattered in all directions from drops and power absorbed by the raindrops when it is radiated by a plane wave of unit power per unit area.

II. METHODOLOGY

A. Data Description and Measurements

Disdrometer measurements used for the modeling and comparisons in this paper were gathered and processed from the J-W RD-80 disdrometer installed in September 2008 at the rooftop of the School of Electrical, Electronic and Computer Engineering, University of KwaZulu-Natal, Durban. A disdrometer measures the raindrop size distribution at a point along the radio path, based on an electro-dynamical receptor which changes the impulse of a falling raindrop into an electrical signal. The sampling time \( T \) of the disdrometer is 60 seconds with the sampling area of 50cm\(^2\)(0.005m\(^2\)). The data was sorted and classified into different types of rain based on rainfall rates \( R \) in millimetre per hour according to [2, 8] as: drizzle (0 mm/hr ≤ \( R < 5 \text{mm/hr} \)), widespread, (5 mm/hr ≥ \( R < 10 \text{mm/hr} \)), shower, (10 mm/hr ≤ \( R < 40 \text{mm/hr} \)) and thunderstorm (\( R > 40 \text{mm/hr} \)). Over 80,000 data samples were obtained from the
disdrometer over a period of 27 months (October 2008 to December 2010). The minimum and maximum rainfall rates were 0.003 mm/hr and 117.148 mm/hr respectively. It should be mentioned that all sum of drops less than 10 in the data samples were removed (ignored). A total of 58,753 data samples were used for analysis. A summary of this is provided in Table I.

The raindrop size distribution is computed from the disdrometer data as:

\[ N(D_j) = \frac{n_j \times 10^6}{v(D_j) \times T \times 5 \times \Delta D_j} \]  \[ \text{[m}^{-3}\text{mm}^{-1}] \]  \[ (3) \]

where \( n_j \) is the number of drops per channel, \( T \) is the one-minute sampling time is 60s, \( S \) is the collecting area of the disdrometer is given as 0.005 m².

### TABLE I: SAMPLED DATA

<table>
<thead>
<tr>
<th>Types of Rainfall</th>
<th>Number of Samples</th>
<th>Total Sum of Drops</th>
<th>Observed Maximum Number of Drops Per Sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drizzle</td>
<td>57,006</td>
<td>12,798,104</td>
<td>2,532</td>
</tr>
<tr>
<td>Widespread</td>
<td>1,114</td>
<td>900,906</td>
<td>2,458</td>
</tr>
<tr>
<td>Shower</td>
<td>581</td>
<td>471,112</td>
<td>2,351</td>
</tr>
<tr>
<td>Thunderstorm</td>
<td>53</td>
<td>59,833</td>
<td>1,819</td>
</tr>
</tbody>
</table>

#### B. The Root Mean Square Error

The Root Mean Square Error (RMSE) is expressed as:

\[ \text{RMSE} = \frac{1}{n} \sum_{j=1}^{n} [\hat{F}(X_j) - F(X_j)]^2 \]  \[ (4) \]

where \( \hat{F}(X_j) \) is the measured data obtained from the disdrometer using (3), \( F(X_j) \) is the modelled data and \( n \) is the number of classes or channels considered.

### III. RECENT RAINDROP SIZE DISTRIBUTION MODELS

One of the earliest works on raindrops where the median diameter \( D_m \) for the drops was determined empirically and given in the form of (5) was carried out by Laws and Parsons (L-P) [9]. It should be noted that L-P worked on the size of raindrops but did not find the RDSD.

\[ D_m = 1.238 R^{0.182} \text{[mm]} \]  \[ (5) \]

Notable and perhaps the first analytical description of the RDSD was carried out by Marshal and Palmer [3]. This has been globally accepted by scientists for the modeling of RDSD for average rain. Their proposed model is given by equation (6) in the form:

\[ N(D) = N_0 \exp(-\Lambda D) \quad (0 \leq D \leq D_{\text{max}}) \]  \[ (6) \]

\( N_0 \) is derived from:

\[ N_0 = \frac{4}{3\pi} \int N(D)D^3v(D)dD \]  \[ (7) \]

where \( \Lambda \) is a constant that tends to increase with rain rates as expressed by the power law of (8) as:

\[ \Lambda = 4.1 R^{-0.21} \text{[mm}^{-1}] \]  \[ (8) \]

\( N_0 \) is usually very close to that of Laws and Parsons given as \( N_0 = 8000 \text{mm}^{-1}\text{m}^{-3} \).

Joss, et al, while measuring the RDSD using the disdrometer in Switzerland obtained a similar exponential form of (6). Table II shows the RDSD parameters of the different types of rain (drizzle, widespread, showers and thunderstorm) with the Joss model in Fig.1 obtained from [10].

### TABLE II: JOSS et al. DSD CONSTANTS

<table>
<thead>
<tr>
<th>Types of Rainfall</th>
<th>( N_\infty \text{ m}^{-3}\text{mm}^{-1} )</th>
<th>( \Lambda \text{ mm}^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drizzle (J-D)</td>
<td>30000</td>
<td>5.7 R^{-0.21}</td>
</tr>
<tr>
<td>Widespread (J-W)</td>
<td>7000</td>
<td>4.1 R^{-0.21}</td>
</tr>
<tr>
<td>Shower (J-S)</td>
<td>8000</td>
<td>4.1 R^{-0.21}</td>
</tr>
<tr>
<td>Thunderstorm (J-T)</td>
<td>1400</td>
<td>3.0 R^{-0.21}</td>
</tr>
</tbody>
</table>

The gamma distribution was initially proposed by Atlas and Ulbrich. It is expressed in the form [11]:

\[ N(D) = N_0 D^\eta \exp(-\Delta D) \quad \text{[m}^{-3}\text{mm}^{-1}] \]  \[ (9) \]

\( No, \Lambda \) and \( \mu \) are parameters to be determined through the measured DSD.

This distribution is particularly useful in tropical climate regions where the exponential distribution has been found on numerous occasions to be inadequate (see, for example [8]).

The Weibull distribution, named after Waloddi Weibull was first proposed by Sekine and Lind [12] as \( N(D) \):

\[ N(D) = N_\infty \left( \frac{D}{\sigma} \right)^{\eta-1} \exp\left(-\left(\frac{D}{\sigma}\right)^{\eta}\right) \text{[m}^{-3}\text{mm}^{-1}] \]  \[ (10) \]

where \( \eta \) and \( \sigma \) are functions of the precipitation rate and the constants \( N_\infty = 1000 \text{ mm}^{-2} \), \( \eta = 0.95 \text{ R}^{0.14} \) and \( \sigma = 0.26 \text{R}^{0.44} \). The Weibull distribution is retained for microwave applications for drizzle, widespread, and shower rain cases. It is also an adequate tool for rain attenuation calculation for microwaves applications [12].

The lognormal model is expressed in the normalised form as [13]:

\[ n(D) = \frac{N(D)}{\int_0^\infty N(D)dD} = \frac{N_T}{\sigma D \sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(D) - \mu}{\sigma}\right)^2\right) \]  \[ (11) \]

where \( N_T \) is the total number of raindrops for all sizes which depends on climates, geographical location of measurements.
and rainfall type, \( \mu \) is the mean of \( \ln(D) \), \( \sigma \) is the standard deviation. This model was proposed primarily for the tropical rainfall. The three parameters are related to rain rate by the expressions in (12) as:

\[
N_T = a_0 R^{b_0} \\
\mu = A_\mu + B_\mu \ln R \\
\sigma^2 = A_\sigma + B_\sigma \ln R
\]  

(12)

where \( a_0, b_0, A_\mu, B_\mu, A_\sigma, B_\sigma \) are coefficients of moment regression. Modeling these parameters as functions of rain rate \( R \) may be inadequate sometimes as they assume different values even at the same rain rate.

**A. Method of Moments**

The method of moments uses the definition of the RDSD moments. For the \( n \)-th moment, \( M_n \) is adopted from [4] as:

\[
M_n = \int_0^\infty D^n N(D) dD
\]  

(13)

For the three-parameter lognormal model, it is expressed as:

\[
M_n = N_T \exp \left[ n \left( \mu + \frac{1}{2} n \sigma^2 \right) \right]
\]  

(14)

Equation (14) solves to give the input parameters \( N_T, \mu, \sigma^2 \) according to [5] as:

\[
N_T = \exp \left[ \frac{24L_3 - 27L_4 + 6L_6}{3} \right] \ 
\mu = \frac{-10L_3 + 13.5L_4 - 3.5L_6}{3} \ 
\sigma^2 = \frac{2L_3 - 3L_4 + L_6}{3}
\]  

(15)\( \quad \) (16)\( \quad \) (17)

\( L_3, L_4 \) and \( L_6 \) are considered the natural logarithms of the measured moments \( M_3, M_4 \) and \( M_6 \). The following input parameters are generated for the lognormal distribution with relation to rain rates as expresses by (11) and (12).

\[
\mu = -0.3104 + 0.1331 \ln R \\
\sigma^2 = 0.0738 + 0.0099 \ln R
\]  

(18)

\[
N_T = 268.07 R^{0.4068}
\]  

**B. Maximum Likelihood Estimation**

This statistical method of estimation originally due to Gauss was developed by Fisher [1]. The method maximizes the likelihood or the product of the density functions of a sample. Generally, the measured raindrop size distribution can be estimated from:

\[
N(D) = PDF \times N_T(D)
\]  

(19)

In this write-up, the PDF for the lognormal is given as:

\[
f(D_i | \mu, \sigma^2) = \left( 2\pi\sigma^2 \right)^{-1/2} D_i^{-1} \exp\left[ \frac{- (\ln(D_i) - \mu)^2}{2\sigma^2} \right] \quad \sigma^2 > 0
\]  

(20)

\( (i = 1, 2 \ldots n) \)

where \( n \) is the sample size. Taking the product of the probability densities of the individual \( D_i \):

\[
L(D_i | \mu, \sigma^2) = \prod_{i=1}^{n} f(D_i | \mu, \sigma^2)
\]  

(21a)

\[
= \prod_{i=1}^{n} \left( 2\pi\sigma^2 \right)^{-1/2} D_i^{-1} \exp\left[ \frac{- (\ln(D_i) - \mu)^2}{2\sigma^2} \right]
\]  

(21b)

IV. METHODS OF RDSD PARAMETERS ESTIMATION

Parameters estimation is a fundamental problem in data analysis. A brief discussion of the method of moments as was used by Ajayi and Olsen [4] and the maximum likelihood estimation techniques as was used by Cohen et al. [19] for estimating the parameters required for modeling is given in this section. The input parameters \( \mu, \sigma \) and \( N_T \) were estimated from the dataset using the two methods (MoM and MLE).
\begin{equation}
= (2\pi \sigma^2)^{-n/2} \prod_{i=1}^{n} D_i^{-1} \exp \left( \sum_{i=1}^{n} \frac{-\ln(D_i) - \mu^2}{2\sigma^2} \right) \tag{21c}
\end{equation}

Taking the natural log of the likelihood function in (21c), we determine \( \hat{\mu} \) and \( \hat{\sigma} \), which maximizes \( L(D_i|\mu, \sigma^2) \). This is done by taking the gradient of the log-likelihood function, \( L \) with respect to \( \mu \) and \( \sigma^2 \) and setting it to zero.

\[
\frac{\partial L}{\partial \mu} = \frac{\sum_{i=1}^{n} \ln(D_i)}{\hat{\sigma}^2} - \frac{n \hat{\mu}}{\hat{\sigma}^2} = 0
\]

Similarly,

\[
\frac{\partial L}{\partial \sigma^2} = -\frac{n}{2 \hat{\sigma}^2} - \frac{\sum_{i=1}^{n} (\ln(D_i) - \hat{\mu})^2}{2 (-\hat{\sigma}^2)^2} = 0
\]

From (22) and (23), the maximum likelihood estimators are given in (24) as:

\[
\hat{\mu} = \frac{\sum_{i=1}^{n} \ln(D_i)}{n}
\]

and

\[
\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} \left( \ln(D_i) - \frac{\sum_{i=1}^{n} \ln(D_i)}{n} \right)^2}{n}
\]

For the lognormal model, using the MLE technique the parameters estimates with relation to rain rates are given in (26):

\[
\hat{\mu} = -0.2189 + 0.1259 \ln R
\]

\[
\hat{\sigma}^2 = 0.2206 + 0.449 \ln R
\]

\[
N_{\text{RDSD}} = 255.9 R^{0.5502}
\]

V. RESULTS AND DISCUSSIONS

A. Comparisons of fit parameters with other DSD Models

Tables IV (a and b) show the parameters estimates for the different types of rainfall using the MoM and the MLE techniques respectively. The measured data was compared with different rainfall types. The selected rainfall rates in mm/hr are 3.47 (drizzle), 8.20 (widespread), 16.78 (shower) and 84.76 (thunderstorm). These are shown in Figs. 2(a) to 2(d) with their probability density functions. In Figs. 2(a) and 2(c), the lognormal model under-estimates the measured data at the lower diameter band for all rain rates while the drizzle has the highest peak with the thunderstorm, drizzle widespread and shower having the least peak in the order for both the MoM and MLE techniques as shown in Figs. 2(b) and 2(d). Thunderstorm has the least occurrence as drizzle dominates Durban for most part of the annual rainfall experienced. The same observation is made in Table I.

B. Root Mean Square Error

To further compare these two methods, we calculated the root mean square errors from (4) and compared them with the measured RDSD. Table III shows the RMSE for the two methods with the MLE having a larger deviation. This deviation can be observed in Figs. 3(a) and 3(b) for rain rates of 5mm/hr and 120mm/hr respectively. Lastly, our proposed lognormal model was compared with similar tropical countries at rain rate of 120 mm/hr as shown in Fig. 4 with the parameter estimates of the RDSD as compared with other DSDs. The proposed model covers a wider distribution up to raindrop diameters above 5mm. Generally, the model compares favourably well with countries within the tropical region.

TABLE III: RMSE FOR MoM AND MLE

<table>
<thead>
<tr>
<th>RAIN RATES (mm/hr)</th>
<th>RMSE MoM</th>
<th>RMSE MLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.1104</td>
<td>3.8408</td>
</tr>
<tr>
<td>10</td>
<td>0.0609</td>
<td>7.6318</td>
</tr>
<tr>
<td>20</td>
<td>0.2099</td>
<td>12.5480</td>
</tr>
<tr>
<td>40</td>
<td>0.0752</td>
<td>22.0287</td>
</tr>
<tr>
<td>60</td>
<td>0.6068</td>
<td>37.2215</td>
</tr>
</tbody>
</table>

TABLE IV: ESTIMATED PARAMETERS OF LOGNORMAL MODEL IN DURBAN

a. MoM

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Drizzle</th>
<th>Widespread</th>
<th>Shower</th>
<th>Thunderstorm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>284.81</td>
<td>342.25</td>
<td>452.27</td>
<td>41.461</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>0.4301</td>
<td>0.1085</td>
<td>-0.095</td>
<td>0.6178</td>
</tr>
<tr>
<td>( A_{\mu} )</td>
<td>-0.3312</td>
<td>-0.3981</td>
<td>-0.4794</td>
<td>0.2183</td>
</tr>
<tr>
<td>( A_{\sigma} )</td>
<td>0.1252</td>
<td>0.2431</td>
<td>0.2952</td>
<td>0.0875</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.0755</td>
<td>0.0784</td>
<td>0.0725</td>
<td>0.0621</td>
</tr>
<tr>
<td>( \beta_{\sigma} )</td>
<td>0.00106</td>
<td>-0.0008</td>
<td>0.0049</td>
<td>0.0085</td>
</tr>
</tbody>
</table>

b. MLE

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Drizzle</th>
<th>Widespread</th>
<th>Shower</th>
<th>Thunderstorm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>266.78</td>
<td>263.3</td>
<td>458.11</td>
<td>225.9</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>0.5662</td>
<td>0.4225</td>
<td>0.1222</td>
<td>0.3871</td>
</tr>
<tr>
<td>( A_{\mu} )</td>
<td>-0.224</td>
<td>-0.2214</td>
<td>-0.3605</td>
<td>-0.6496</td>
</tr>
<tr>
<td>( B_{\mu} )</td>
<td>0.1239</td>
<td>0.1339</td>
<td>0.209</td>
<td>0.2755</td>
</tr>
<tr>
<td>( A_{\sigma} )</td>
<td>0.213</td>
<td>0.1691</td>
<td>0.1647</td>
<td>0.8082</td>
</tr>
<tr>
<td>( B_{\sigma} )</td>
<td>0.0423</td>
<td>0.0897</td>
<td>0.1024</td>
<td>-0.083</td>
</tr>
</tbody>
</table>

TABLE V: LOGNORMAL PARAMETERS FOR SELECTED TROPICAL COUNTRIES

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Nigeria</th>
<th>Malaysia</th>
<th>India</th>
<th>Singapore</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>108</td>
<td>45.325</td>
<td>546</td>
<td>276.18</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>0.363</td>
<td>0.6703</td>
<td>0.469</td>
<td>0.3815</td>
</tr>
<tr>
<td>( A_{\mu} )</td>
<td>-0.195</td>
<td>-0.3914</td>
<td>-0.538</td>
<td>-0.4286</td>
</tr>
<tr>
<td>( b_{\sigma} )</td>
<td>0.199</td>
<td>0.1873</td>
<td>0.017</td>
<td>0.1458</td>
</tr>
<tr>
<td>( A_{\sigma} )</td>
<td>0.137</td>
<td>0.4072</td>
<td>0.0689</td>
<td>0.1564</td>
</tr>
<tr>
<td>( B_{\sigma} )</td>
<td>0.013</td>
<td>-0.0586</td>
<td>0.076</td>
<td>0.00913</td>
</tr>
</tbody>
</table>
VI. CONCLUSION

In this paper, two methods of parameter estimation for the lognormal distribution model (MoM and MLE) are compared. The lognormal distribution model is seen to be adequate for the modeling of (rain) drop size distribution (DSD) in the region. The model compares favourably well with DSD model within the tropical region. The MLE technique gave a larger deviation error from the measured data while the MoM revealed a fairly better agreement with the measured data. However, an optimization of the $N_T$ parameter especially for the MLE technique should be studied further. This work is important to estimate the specific attenuation.

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REFERENCE


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