A bandwidth market approach to resource pricing

A.E. Krzesinski and G.-A. Lusilao-Zodi

***Abstract*— Consider a path-oriented telecommunications network where packets arrive to each route in a Poisson process and the packet lengths are exponentially distributed. Each route can queue a finite number of packets while one packet is being transmitted. Each accepted packet generates an amount of revenue for the route manager. At specified time points a route manager can acquire additional bandwidth in order to carry more packets and earn more revenue; alternatively a route manager can earn additional revenue by selling surplus bandwidth to other route managers that (possibly temporarily) value it more highly. We present a method for efficiently computing the buying and the selling prices of bandwidth.**

I. Introduction

We consider a bandwidth management scheme in which each route places a value on bandwidth, dependent on its current bandwidth assignment and its current occupancy. Under our proposed scheme, a bandwidth manager is assigned to each route. Each manager calculates the revenue that the route would gain should the route acquire an extra unit of bandwidth (the "buying price") and also the revenue that the route would lose should the route give up a unit of bandwidth (the "selling price"). The manager then uses these prices to determine whether it should acquire or release a unit of bandwidth or do neither. The bandwidth prices form the basis for a mechanism to re-allocate bandwidth from routes that place a low value on bandwidth to routes that place a high value on bandwidth. The question arises as to how these prices should be calculated.

We present a bandwidth pricing model that is based on the approach presented in [2]. However, we use an $M/M/1/K$ queueing system to model the process of placing a value on bandwidth. We consider the buying and selling of bandwidth in terms of the variation in the lost revenue when the packet service rate increases or decreases. Revenue is lost when packets are dropped. The lost revenue can be controlled by varying the bandwidth allocated to the route. For each unit of bandwidth bought the packet service rate will increase, the loss probability will decrease and the rate of earning revenue will increase. Conversely for each unit of bandwidth sold the packet service rate will decrease, the loss probability will increase and the rate of earning revenue will decrease.

The paper is organised as follows. We first define a model to compute the expected lost revenue. We next develop a recursive formula for the efficient computation of the lost revenue. Finally we use the expected lost revenue to derive the buying and selling prices of a unit of bandwidth.

A.E. Krzesinski and G.-A. Lusilao-Zodi, Department of Mathematical Sciences, University of Stellenbosch, 7600 Stellenbosch, South Africa. Email: aek1,glusilaozodi@cs.sun.ac.za. Tel: +27 21 8084232 Fax: +27 21 8084416

This work is supported by grant numbers 2047362 and 2677 from the South African National Research Foundation, Siemens Telecommunications and Telkom SA Limited.

II. Model and analysis

The $M/M/1/K$ queueing system can be modelled by a continuous time Markov chain with state space $\{0, 1, \ldots, K\}$ and transition rates

$$q_{n,n+1} = \begin{cases} \lambda & 0 \leq n < K \\ 0 & n = K \end{cases}$$
$$q_{n,n-1} = \begin{cases} \mu & 0 < n \leq K \\ 0 & n = 0 \end{cases}$$

where λ and μ are positive constants denoting respectively the mean arrival rate and the mean service rate of a packet. If a new packet arrives and there are already K packets in the system, the new packet is lost: the packet is dropped from the system and never returns.

Let θ denote the expected revenue generated per accepted packet. Let $R_n(t)$ denote the expected lost revenue in the interval $[0, t]$ given that there are n packets in the $M/M/1/K$ system at time 0. The quantity t is referred to as the planning horizon. Let $R_n(t|x)$ be the same quantity conditional on the fact that the first time that the $M/M/1/K$ queue departs from the state n is x. Then

$$R_n(t|x) = \begin{cases} 0 & 0 \leq n < K, t < x \\ \theta\lambda & n = K, t < x \\ \dfrac{\mu}{\lambda+\mu} R_{n-1}(t-x) & \\ + \dfrac{\lambda}{\lambda+\mu} R_{n+1}(t-x) & 0 < n < K, t \geq x \\ \theta\lambda x + R_{K-1}(t-x) & n = K, t \geq x. \end{cases}$$

Let $F_n(x)$ be the distribution of time x until the first transition, when there are n packets in the system. $F_n(x)$ is exponential with parameter $\lambda + \mu$. There are three cases to be considered.

Case 1: $n = 0$. In this case $dF_0(x) = \lambda e^{-\lambda x} dx$ and

$$R_0(t) = \int_0^t R_0(t|x) dF_0(x) = \int_0^t R_1(t-x) \lambda e^{-\lambda x} dx.$$

Case 2: $0 < n < K$. In this case $dF_n = (\lambda + \mu) e^{-(\lambda+\mu)} dx$

and

$$\begin{aligned} &R_n(t) \\ &= \int_0^t R_n(t|x)dF_n(x) \\ &= \int_0^t \left(\frac{\mu}{\lambda+\mu}R_{n-1}(t-x) + \frac{\lambda}{\lambda+\mu}R_{n+1}(t-x)\right) \\ &\quad (\lambda+\mu)e^{-(\lambda+\mu)x}dx \\ &= \int_0^t (\mu R_{n-1}(t-x) + \lambda R_{n+1}(t-x))e^{-(\lambda+\mu)x}dx. \end{aligned}$$

Case 3: $n = K$. In this case $dF_K(t) = \mu e^{-\mu}dx$ and

$$\begin{aligned} &R_K(t) \\ &= \int_0^t R_K(t|x)dF_K(x) + \int_t^\infty R_K(t|x)dF_K(x) \\ &= \mu\int_0^t (R_{K-1}(t-x) + \theta\lambda x)\, e^{-\mu x}dx \\ &\quad +\mu\int_t^\infty \theta\lambda t e^{-\mu x}dx \\ &= \mu\int_0^t R_{K-1}(t-x)e^{-\mu x}dx + \frac{\theta\lambda}{\mu}(1-e^{-\mu}). \end{aligned}$$

Taking the Laplace transform of the above three equations we obtain

$$\widetilde{R}_0(s) = \frac{\lambda}{s+\lambda}\widetilde{R}_1(s) \qquad (1)$$

$$\widetilde{R}_n(s) = \frac{\lambda}{s+\mu+\lambda}\widetilde{R}_{n+1}(s) + \frac{\mu}{s+\mu+\lambda}\widetilde{R}_{n-1}(s) \qquad 0 < n < K \quad (2)$$

$$\widetilde{R}_K(s) = \frac{1}{s+\mu}\left(\mu\widetilde{R}_{K-1}(s) + \frac{\theta\lambda}{s}\right). \qquad (3)$$

From Eqn. (2) and using the methods presented in [2] we obtain the recurrence relation

$$P_{n+1}(\xi) = (\xi + \mu/\lambda + 1)P_n(\xi) - (\mu/\lambda)P_{n-1}(\xi) \qquad (4)$$

for $n \geq 1$ where $\xi = s/\lambda$.

The next step is to express Eqn. (4) in terms of orthogonal polynomials. First substitute

$$P_n(\xi) = Q_n(\xi + \mu/\lambda + 1)$$

so that Eqn. (4) becomes

$$\begin{aligned} Q_{n+1}(\xi+\mu/\lambda+1) &= (\xi+\mu/\lambda+1)\; Q(\xi+\mu/\lambda+1) \\ &- (\mu/\lambda)\; Q_{n-1}(\xi+\mu/\lambda+1) \end{aligned}$$

which can be written as

$$Q_{n+1}(\phi) = \phi Q(\phi) - (\mu/\lambda)Q_{n-1}(\phi) \qquad (5)$$

where $\phi = \xi+\mu/\lambda+1$. Next let α be a constant (to be chosen later) and let

$$S_n(\phi) = \alpha^n Q_n(\phi).$$

Eqn. (5) becomes

$$\frac{1}{\alpha^{n+1}}S_{n+1}(\phi) = \frac{\phi}{\alpha^n}S_n(\phi) - \frac{\mu}{\lambda}\frac{1}{\alpha^{n-1}}S_{n-1}(\phi).$$

Multiplying throughout by α^{n+1} yields

$$S_{n+1}(\phi) = \alpha\phi S_n(\phi) - (\mu/\lambda)\alpha^2 S_{n-1}(\phi).$$

Now choose α such that $(\mu/\lambda)\alpha^2 = 1$. Then

$$S_{n+1}(\phi) = \alpha\phi S_n(\phi) - S_{n-1}(\phi)$$

which can be written as

$$S_{n+1}\left(\frac{2}{\alpha}\frac{\alpha\phi}{2}\right) = 2\left(\frac{\alpha\phi}{2}S_n\left(\frac{2}{\alpha}\frac{\alpha\phi}{2}\right)\right) - S_{n-1}\left(\frac{2}{\alpha}\frac{\alpha\phi}{2}\right). \qquad (6)$$

Let $x = \alpha\phi/2$ and define

$$S_n(2x/\alpha) = C_n(x)$$

so that Eqn. (6) becomes

$$C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x)$$

which describes the Chebychev polynomials.

To obtain the explicit form of $C_n(x)$, we express $C_n(x)$ in term of $P_n(x)$

$$\begin{aligned} C_n(\xi) &= S_n(2\xi/\alpha) \\ &= \alpha^n Q_n(2\xi/\alpha) \\ &= \alpha^n P_n(2\xi/\alpha - \mu/\lambda - 1) \end{aligned} \qquad (7)$$

where $n \geq 1$ and $\alpha^2 = \lambda/\mu$. Using Eqn. (1) we obtain

$$P_1(\xi) = (\xi+1)P_0(\xi).$$

Taking $P_0(x) = 1$ from Eqn. (7) and Eqn. (1) it can be shown that

$$C_1(\xi) = 2\xi - \alpha\mu/\lambda.$$

Using the initial conditions $C_0(\xi) = 1$ and $C_1(\xi) = 2\xi - \alpha\mu/\lambda$ it is shown in [4] pp 204, that

$$C_n(\xi) = 2T_n(\xi) + U_{n-2}(\xi) - (\alpha\mu/\lambda)U_{n-1}(\xi)$$

where $T_n(\xi)$ and $U_n(\xi)$ are respectively the first and the second kind of Chebychev polynomials. The solution of the recurrence relation is

$$P_n(\xi) = \alpha^{-n}C_n(\alpha(\xi+\mu/\lambda+1)/2)$$

where $n \geq 0$, $\alpha^2 = \lambda/\mu$ and the $C_n(\cdot)$ are Chebychev polynomials.

Using the same arguments present in [2], it follows that the solution of Eqns. (1) through (3) is given by $\widetilde{R}_n(s) = A(s)P_n(s/\lambda)$, where $P_n(s/\lambda)$ are Chebychev polynomials. Using the condition for $n = K$, we obtain

$$A(s) = \left(\frac{1}{s}\right)\left(\frac{\theta\lambda}{(s+\mu)P_K(s/\lambda) - \mu P_{K-1}(s/\lambda)}\right) \quad (8)$$

and so

$$\widetilde{R}_n(s) = \left(\frac{1}{s}\right)\left(\frac{\theta\lambda}{(s+\mu)P_K(s/\lambda) - \mu P_{K-1}(s/\lambda)}\right) P_n(s/\lambda). \quad (9)$$

III. A NUMERICALLY STABLE CALCULATION OF $\widetilde{R}_n(s)$

A numerically stable computation of $\widetilde{R}_n(s)$ is derived as follows [3]. We first compute $\widetilde{R}_K(s)$

$$\lambda\widetilde{R}_K(s) = \frac{\theta/x}{x + \varrho - \varrho\dfrac{P_{K-1}(x)}{P_K(x)}}$$

where $x = s/\lambda$ and $\varrho = \mu/\lambda$. From Eqn. (2)

$$\frac{P_{n-1}(x)}{P_n(x)} = \frac{1}{F - \varrho\dfrac{P_{n-2}(x)}{P_{n-1}(x)}}$$

for $0 < n < K$ where $F = x + \varrho + 1$. From Eqn. (1), the recursion is terminated by

$$\frac{P_0(x)}{P_1(x)} = \frac{1}{x+1}.$$

We can now compute $\widetilde{R}_n(s)$

$$\widetilde{R}_n(s) = \widetilde{R}_K(s)\prod_{i=n+1}^{K}\frac{P_{i-1}(x)}{P_i(x)}$$

where $0 \leq n < K$. The successive terms are bounded $0 < \left|\frac{P_{i-1}(x)}{P_i(x)}\right| < 1$ where $|z|$ denotes the norm of the complex variable z.

In order to derive $R_n(t)$, we need to invert $\widetilde{R}_n(s)$ which can be done using the Euler method presented in [1].

IV. NUMERICAL EXAMPLES

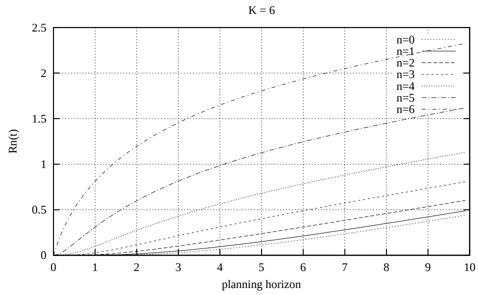
This section presents several examples of the computation of the lost revenue function $R_n(t)$.

A. The $M/M/1/6$ queueing system

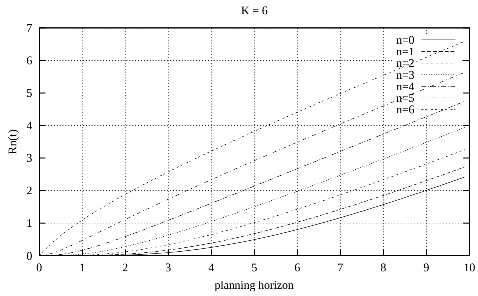
Fig 1(a) presents the lost revenue function $R_n(t)$ for a small $M/M/1/K$ queue where $K = 6$, $\theta = 1$, $\lambda = 1.5$, $\mu = 2$, $n = \{0, \ldots, 6\}$ and the planning horizon $t \in [0, 10)$. The function $R_0(t)$ is the lowest curve and the function $R_6(t)$ is the highest curve. We observe that $R_n(t)$ is increasing with n. We also observe that with increasing t, $R_n(t)$ is well approximated by a linear function with a slope equal to $\theta\lambda P(\rho, K)$ where $\rho = \lambda/\mu$ and

$$P(\rho, K) = \rho^K \frac{1-\rho}{1-\rho^{K+1}} \quad (10)$$

is the equilibrium blocking probability that K packets are in the system. The difference in the height of the linear part of the functions $R_{n+1}(t)$ and $R_n(t)$ reflects the difference in the expected revenue incurred after equilibrium is reached when the system starts with $n+1$ packets rather than n packets.



(a) $\mu = 2$



(b) $\mu = 1$

Fig. 1. The $M/M/1/K$ lost revenue function $R_n(t)$ for $n = 0, \ldots, 6$, $K = 6$ and $\lambda = 1.5$.

Fig. 1(a) presents the lost revenue function for a system with low blocking ($P(\rho, K) = 0.05$). Fig. 1(b) presents the lost revenue function for a system with a larger blocking which is achieved by decreasing μ to 1. The blocking probability $P(\rho, K)$ is equal to 0.35.

The load and hence the equilibrium slope of the curves, is much greater in Fig. 1(b) than in Fig. 1(a). However the latter is still given by $\theta\lambda P(\rho, K)$. The difference in the equilibrium heights of the function $R_{n+1}(t)$ and $R_n(t)$ does not vary as much between $n = 0$ and $n = 5$ as for the low blocking system. This reflects the fact that in the low blocking system, states with high occupancy are unlikely to be visited in the short term if the route does not start with a high occupancy. In the high blocking system system, the probability of moving to states with high occupancy in the short term is relatively higher even if the starting state has a low occupancy [2].

B. The $M/M/1/100$ queueing system

Fig. 2 presents the the lost revenue function $R_n(t)$ for a larger system with $K = 100$, $\theta = 1$, $\lambda = 85$, $\mu = 80$ and $n = \{0, 25, 50, 75, 90, 100\}$. As for the small system, we observe that $R_n(t)$ is increasing with n and we also observe that after the initial period in which the starting state has an effect, the $R_n(t)$ increase linearly at the same rate. The $R_n(t)$ increase with increasing n, with more a more pronounced increase as n becomes large.

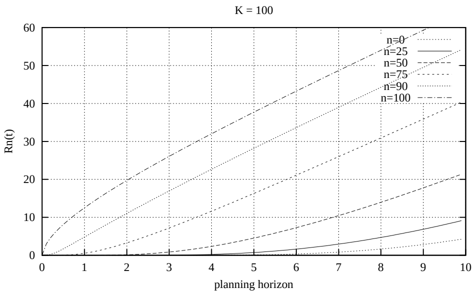


Fig. 2. The $M/M/1/K$ lost revenue function $R_n(t)$ for $n = \{0, 25, 50, 75, 90, 100\}$, $K = 100$, $\lambda = 85$ and $\mu = 80$.

V. The Price of Bandwidth

The expected lost revenue is transformed into a price at which one unit of extra bandwidth (one unit of bandwidth is equal to Δ bps) should be "bought" or "sold". We assume that the network manager is making buying and selling decisions for a planning horizon of T time units, and that the choice of T is the decision of the network manager.

As in [2], once the manager has chosen T, we regard the value of an extra unit of bandwidth as the difference in the total expected lost revenue over the time interval $[0, T]$ if the system were to increase the packet service rate by one unit at time zero. Conversely, we calculate the selling price of unit of bandwidth as the difference in the total expected lost revenue over time $[0, T]$ if the system were to decrease the packet service rate by one unit.

The buying $B_n(T)$ and the selling $S_n(T)$ prices of bandwidth when n packets are present at the route (1 packet is in service, $n-1$ packets are queued), the route waiting line has capacity $K-1$, the mean packet service rate is μ and the planning horizon is T can be written as

$$\begin{aligned} B_n(T) &= R_{n,\mu,K}(T) - R_{n,\mu+\Delta,K}(T) \\ S_n(T) &= R_{n,\mu-\Delta,K}(T) - R_{n,\mu,K}(T) \end{aligned}$$

where $R_{n,\mu,K}(T)$ is the expected lost revenue. We expect that for all n, K and T, $S_n(T) > B_n(T)$. Some examples of $B_n(T)$ and $S_n(T)$ are given in the following section.

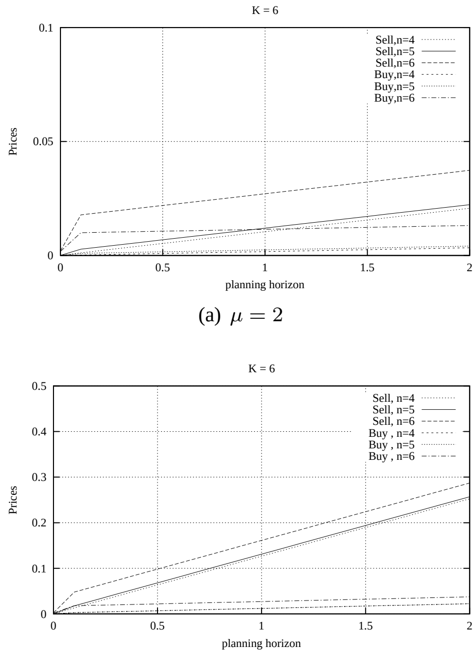
A. The $M/M/1/6$ queueing system

Fig. 3(a) and Fig. 3(b) present the buying and selling prices for a $M/M/1/K$ system with $K = 6$, $\theta = 1$, $\lambda = 3500$, $n \in \{4, 5, 6\}$ in the case of low and high blocking respectively .

The figures show that the selling price $S_n(T)$ is greater than the buying price $B_n(T)$ for all n and T. As n approaches the route capacity K the system places a higher value on the available bandwidth for both the buying and the selling prices.

B. The $M/M/1/50$ queueing system

Similar observations can be made for a larger system with $K = 50$, $\lambda = 95000$ and $\mu = 100000$.



(a) $\mu = 2$

(b) $\mu = 1$

Fig. 3. The $M/M/1/6$ buying and selling prices.

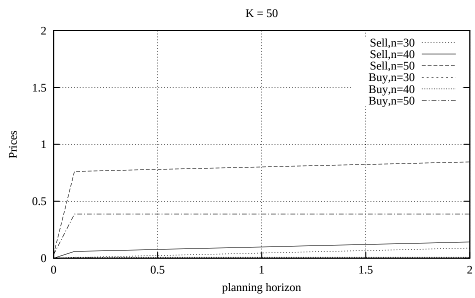


Fig. 4. The $M/M/1/50$ buying and selling prices for $n \in \{30, 40, 50, \}$, $K = 50$, $\lambda = 95000$ and $\mu = 100000$.

VI. Conclusion

This paper presents a model which, given the current state of a $M/M/1/K$ queue, computes the expected lost revenue $R_n(t)$ due to packet loss. Assuming a planning horizon t, we translate the expected losses into buying and selling prices of one unit of bandwidth.

The expected lost revenue $R_n(t)$ is expressed as a system of renewal equations. We then derived a system of recurrence relations satisfied by the Laplace Transform of $R_n(t)$. It was shown that the solution of this system of recurrence relations could be determined in terms of Chebychev polynomials. We inverted these Laplace Transforms numerically using the Euler method.

We demonstrated the computation of these prices for both a small system with $K = 6$, and a larger system with $K = 100$.

We intend to incorporate the $M/M/1/K$ pricing model into a simulation model of a distributed, efficient and scalable

Network Resource Controller (NRC). The NRC computes state-dependent bandwidth prices which form the basis of a bandwidth re-allocation (trading) scheme in connection-oriented networks. The NRC uses the bandwidth prices to optimally adjust (buying/selling) the bandwidth of the network paths within a routing domain. The NRC will also apply resource- and policy-based Call Admission Control (CAC) to provide a quality guaranteed transport service. NRCs will peer with neighbouring NRCs to obtain optimal end-to-end Quality of Service (QoS).

REFERENCES

[1] J. Abate and W. Whitt. Numerical Inversion of Laplace Transforms of Probability Distributions. ORSA Journal on Computing 7:36-43 (1995).

[2] B.A. Chiera and P.G. Taylor. What is a Unit of Capacity Worth? Probability in the Engineering and Informational Sciences, 16:513–522 (2002).

[3] B.A. Chiera, A.E. Krzesinski and P.G. Taylor. Some Properties of the Capacity Value Function. SIAM Journal on Applied Mathematics, Vol 65 No 4 pp 1407–1419 (2005).

[4] T.S. Chihara. An Introduction to Orthogonal Polynomials, New York (1978).

Guy-Alain Lusilao-Zodi graduated from the University of Kinshasa in the DRC with first class honours in Mathematics. He obtained a post-graduate diploma from the African Institute of Mathematical Sciences in 2005. He is currently an MSc student in the Department of Mathematical Sciences at the University of Stellenbosch.