Optimisation of the Pilot-to-Data Power Ratio in the MQAM-Modulated OFDM Systems with MMSE Channel Estimation

Eugene Golovins, and Neco Ventura
{egolovins,neco}@crg.ee.uct.ac.za

Abstract—Modern high-rate wireless OFDM systems use dedicated pilot subcarriers to obtain regular estimates of the channel frequency response and perform equalisation followed by coherent detection on other (data) subcarriers. Boosting power of the pilot subcarriers relative to the data subcarriers can increase channel estimation accuracy. However due to the overall transmit signal power constraint, dictated by energy conservation and restricted service zone radius, it will inevitably lead to degradation of performance of the equaliser-detector couple being a result of the receiver noise level growth on the data subcarriers. In this article we derive an analytical expression for the optimal pilot-to-data power ratio (PDR) providing maximum signal-to-noise power ratio (SNR) at the output of the equaliser and hence improving BER performance. We also examine effects of the optimal and suboptimal minimum mean squared error (MMSE) channel estimator design and their impact on PDR selection.

***Index Terms*—minimum mean squared error, peak-to-average power ratio, pilot-to-data power ratio**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) technology has become increasingly popular in recent years because of its efficiency in reducing the severe effects of frequency-selective fading. Increased transmission rates in OFDM are achieved due to the use of spectrally efficient quadrature amplitude modulation (QAM). However coherent demodulation of the QAM signals requires knowledge or an accurate estimate of the channel frequency response (CFR), in order to minimise the probability of detection error [1]. CFR estimates can be obtained using pilot-symbol assisted modulation (PSAM) where training data is transmitted on certain subcarriers within the OFDM band allowing for CFR measurements with subsequent interpolation onto other subcarriers.

Performance of several frequency-domain interpolators (e.g., linear, trigonometric function based, etc.) is evaluated analytically in [2] for channels with different frequency selectivity characteristics. [3] and [4] propose to use polynomial regression models to approximate CFR in the time and frequency domain. Estimators utilising either statistical or deterministic channel model assumptions are considered to be more robust in comparison with the polynomial interpolation techniques. A frequency-domain linear minimum MSE (LMMSE) CFR estimator has been introduced in [5] with low-rank approximation to reduce complexity. In [6] the two-dimensional optimal MMSE estimator is derived using the separation property of the time-frequency channel correlation function. An alternative to frequency-domain processing is the so-called DFT-methods in which the CFR is transformed into the channel impulse response (CIR) and estimation benefits from the suggested finite CIR length peculiar to multipath channels by filtering out noise in CIR tail part. Least squares (LS) [7], maximum likelihood [8], [9] and LMMSE [7], [11] schemes have been proposed and found to outperform frequency-domain algorithms [7], [8] while being of lesser complexity [10], [11].

In the conditions of the transmitted power constraint, dictated by energy conservation and restricted service zone radius, questions about optimal PSAM scheme arise essentially. One cannot achieve higher signal-to-noise power ratio (SNR) on the pilot subcarriers at the receiver input without sacrificing SNR on the data subcarriers. This trade-off stipulates existence of the optimal pilot structure generally depending on the channel estimation and data detection method. In [12] the optimal pilot subcarrier placement and number have been derived using a lower bound on the average ergodic capacity of the channel as an optimisation criterion. Alternatively, authors of [13] searched analytically for these parameters, as well as for the optimal pilot-to-data power ratio (PDR) to minimise the symbol error rate in the M-PSK modulated OFDM system with receiving diversity.

In this article we derive optimal PDR for the QAM modulated OFDM system with MMSE channel estimation (both ideal and mismatched) and the standard one-tap equalisation. The rest of the work is organised as follows. Section II contains a general OFDM system description. Section III outlines the design of the channel estimator. Section IV contains the derivation of the SNR at the equaliser output and its maximisation to yield the optimum PDR. Section V discusses theoretical and simulation results. A conclusion in Section VI finalises the paper.

II. OFDM System Model

In this work a single-input-single-output discrete-time baseband OFDM model is considered. It includes transmitter, receiver and equivalent discrete-time band-limited channel model. The transmitter and the receiver are assumed to have ideal timing and frequency synchronisation.

In the transmitter, a serial random binary stream is divided into parallel binary streams, each of which passes through the MQAM modulation scheme. The OFDM symbol is formed as the result of an IDFT, applied to N parallel (P training and $N-P$ data) complex-valued modulation subsymbols X_n, $n = 0,...,N-1$. The resultant waveform is converted to a serial sequence of samples. Before transmission each OFDM symbol is prepended with a cyclic prefix (CP), which is a copy of the last portion of the OFDM symbol.

In the considered scenario the channel is assumed to be slowly time-varying, i.e. CFR is approximately constant during one OFDM symbol, so there is no loss of orthogonality between subcarriers. As reported in [9], this assumption holds if duration of the processing block, i.e. OFDM symbol without CP is

$$T \le 0.01/f_{\mathrm{D}}, \quad (1)$$

where $f_{\mathrm{D}} = f_{\mathrm{c}} v/c$ is the maximum Doppler frequency, f_{c} is the RF carrier frequency, v is the speed of relative movement between the transmitter and the receiver, and c is the speed of light.

With the time-invariant channel response assumption, CP length N_{cp} can be made large enough to accommodate a finite CIR h_m, $m = 0,...,L-1$, where the maximum sample-normalised excess delay $L-1 \le N_{\mathrm{cp}}$. Thus, the intersymbol interference (ISI) between consecutive OFDM symbols will be eliminated.

In this paper, to reflect the CIR process we adopt the bandlimited filter model with a strict response energy enclosure within the first L taps with the tap-gains described by the vector $\mathbf{h}(i) = [h_0(i) \ \cdots \ h_{L-1}(i)]^{\mathrm{T}}$, where i denotes the serial index of the OFDM symbol. $\mathbf{h}(i)$ is linked to the commonly used block quasi-static approximation of the wide-sense stationary uncorrelated scattering (WSSUS) K-path response model [6]

$$g(t) = \sum_{k=0}^{K-1} a_k(i)\,\delta(t - i\tau_k) \quad (2)$$

according to the formula

$$\mathbf{h}(i) = \mathbf{\Sigma}\,\mathbf{a}(i), \quad (3)$$

where k is the path index, τ_k is the path delay, path gains $\mathbf{a}(i) = [a_0(i) \ \cdots \ a_{K-1}(i)]^{\mathrm{T}}$ represent zero-mean complex Gaussian variables produced by lowpass-filtered independent stochastic processes, and the elements of the band-limiting matrix $\mathbf{\Sigma}$ are given as $[\mathbf{\Sigma}]_{m,k} = \dfrac{\sin(m - N\tau_k/T)}{m - N\tau_k/T}$. Without the loss of generality hereafter we will consider the average CIR energy to be unity normalised, i.e. $\mathrm{E}[\mathbf{h}(i)^{\mathrm{H}}\mathbf{h}(i)] = 1$.

At the receiver side, after removing the CP and applying DFT to the ith OFDM symbol we get a vector of the received subsymbols:

$$\mathbf{Y}(i) = [Y_0(i) \ \cdots \ Y_{N-1}(i)]^{\mathrm{T}} = \mathbf{X}_{[\mathrm{D}]}(i)\mathbf{H}(i) + \mathbf{W}(i), \quad (4)$$

where $\mathbf{X}_{[\mathrm{D}]}(i) = \mathrm{diag}[\mathbf{X}(i)]$ denotes a diagonal matrix with the data and pilot subsymbols $X_n(i)$, $0 \le n \le N-1$; $\mathbf{H}(i) = [H_0(i) \ \cdots \ H_{N-1}(i)]^{\mathrm{T}}$ is the CFR vector; and $\mathbf{W}(i) = [W_0(i) \ \cdots \ W_{N-1}(i)]^{\mathrm{T}}$ are the DFT-transformed white Gaussian noise (WGN) variables.

Before the parallel subset of the received data subsymbols $Y_{\mathrm{d}_l}(i)$, $l \in [0; N-P-1]$, can be demodulated, it is necessary to correct signal distortions caused by passing through the channel. As OFDM systems work by resolving the frequency domain, a simple block-oriented one-tap equaliser can be used:

$$\hat{X}_{\mathrm{d}_l}(i) = Y_{\mathrm{d}_l}(i) / \hat{H}_{\mathrm{d}_l}(i), \quad (5)$$

where $\hat{H}_{\mathrm{d}_l}(i)$ denotes the CFR estimate on the corresponding data subcarrier obtained using the pilot subsymbols $X_{\mathrm{p}_k}(i)$, $k \in [0; P-1]$.

Substitution of (4) in (5) yields

$$\hat{X}_{\mathrm{d}_l}(i) = X_{\mathrm{d}_l}(i) + \frac{X_{\mathrm{d}_l}(i)V_{\mathrm{d}_l}(i) + W_{\mathrm{d}_l}(i)}{\hat{H}_{\mathrm{d}_l}(i)} = X_{\mathrm{d}_l}(i) + \Psi_{\mathrm{d}_l}(i), \quad (6)$$

where $V_{\mathrm{d}_l}(i) = H_{\mathrm{d}_l}(i) - \hat{H}_{\mathrm{d}_l}(i)$ denotes the CFR estimation error, whereas $\Psi_{\mathrm{d}_l}(i)$ represents a cumulative noise term at the post-equalisation stage.

III. MMSE Channel Estimation

We denote the received subsymbols at the pilot positions as

$$\mathbf{Y^p} = \mathbf{X^p_{[D]}}\mathbf{H^p} + \mathbf{W^p} = \mathbf{X^p_{[D]}}\mathbf{C}\mathbf{H} + \mathbf{W^p} = [Y_{\mathrm{p}_0} \ \cdots \ Y_{\mathrm{p}_{P-1}}]^{\mathrm{T}}, \quad (7)$$

where $\mathbf{X^p_{[D]}} = \mathrm{diag}(\mathbf{X^p})$ contains reference values of P equipowered pilot subsymbols, and $\mathbf{C}$ is the $P \times N$ -size selection matrix with the elements $C_{k,n} = \begin{cases} 1, & \text{if } n = \mathrm{p}_k \\ 0, & \text{otherwise} \end{cases}$ that is needed to extract samples of the CFR $\mathbf{H} = [H_0 \ \cdots \ H_{N-1}]^{\mathrm{T}}$ corresponding to the pilot subcarriers indexed p_k, $0 \le k \le P-1$. (Hereafter OFDM symbol index i is omitted for brevity, as the processing is done on a symbol-by-symbol basis, without considering correlation with the neighbouring symbols.)

Pilot-aided MMSE estimator is given by the equation [11]:

$$\hat{\mathbf{H}} = \tilde{\mathbf{R}}_{\mathbf{HH}}\mathbf{C}^{\mathrm{H}}\left[\mathbf{C}\tilde{\mathbf{R}}_{\mathbf{HH}}\mathbf{C}^{\mathrm{H}} + SNR_{\mathrm{p}}^{-1}\mathbf{I}\right]^{-1}\mathbf{X^{p}_{[D]}}^{-1}\mathbf{Y^p}, \quad (8)$$

where $\tilde{\mathbf{R}}_{\mathbf{HH}}$ is the assumed CFR correlation matrix, which should be equal to the true CFR correlation matrix $\mathbf{R}_{\mathbf{HH}}$ to yield the MMSE; and $SNR_{\mathrm{p}} = \sigma_{\mathrm{p}}^2/\sigma_{\mathrm{w}}^2$ is the signal-to-noise power ratio observed at the equipowered pilot subcarriers, with σ_{p}^2 and σ_{w}^2 being equal to the pilot power and the noise variance respectively. Given constant noise variance, σ_{w}^2, across all subcarriers in the OFDM spectrum, it is easy to show that SNR_{p} is related to the SNR observed at the receiver input according to the following equation:

$$SNR_{\mathrm{p}} = \frac{N\kappa}{P\kappa + N - P}SNR, \quad (9)$$

where $\kappa = \sigma_{\mathrm{p}}^2/\bar{\sigma}_{\mathrm{d}}^2$ represents the average pilot-to-data power ratio. Here the average power per MQAM-modulated data subcarrier, $\bar{\sigma}_{\mathrm{d}}^2$, is proportional to the peak modulation power within the MQAM constellation, $\sigma_{\mathrm{d,max}}^2$, namely $\bar{\sigma}_{\mathrm{d}}^2 = \sigma_{\mathrm{d,max}}^2/PAR$ with the inverse of the proportionality coefficient (also known as the peak-to-average power ratio) being equal for the rectangular-type MQAM constellations to

$$PAR = \frac{3\left(2^{b/2}-1\right)^2}{2^b-1}, \quad (10)$$

where b denotes the number of bits carried by one modulation symbol ($M = 2^b$). For example, for 4QAM (QPSK) with $b = 2$ $PAR = 1$, for 16QAM with $b = 4$ $PAR = 9/5$, for 64QAM with $b = 6$ $PAR = 7/3$, etc.

IV. PDR Optimisation

Prior to the derivation of the optimal PDR, one has to take into account other pilot-related parameters and their recommended settings. These parameters are discussed in Subsection A.

Subsection B addresses the impact of the imperfect channel estimator design on MSE and establishes a robust formulation of the SNR maximisation problem.

A. *Optimal Pilot Arrangement*

It has been found in the previous research [12][13] that the pilot subcarriers should be equally spaced within the OFDM spectrum to achieve optimal system performance with regard to channel estimation. This condition is satisfied when N/P is equal to an integer, then indices of pilot subcarriers in (7) are given as

$$C_{k,n} = \begin{cases} 1, \text{ if } \ n = p_k = kN/P + \nu \\ 0, \text{ otherwise} \end{cases}, \quad (11)$$

where $0 \le \nu \le N/P - 1$

If N/P is not an integer, one can use a suboptimal uniform pilot placement proposed in [13]. Such a scheme was found to perform almost as well as the equispaced one due to the small CFR interpolation error.

Another important parameter is the number of pilot subcarriers P. It is obvious that P should be as small as possible to reduce transmission redundancy. In [12] the minimum possible value of $P = L$ was found to be optimal from the ergodic capacity standpoint. Our previous work [10], [11] also demonstrated that doubling the number of pilots in comparison with the anticipated CIR length value ($P = L$) results only in a small channel estimation accuracy improvement. On the contrary, varying PDR was found to have a much more substantial influence on the system performance.

B. *Derivation of the Channel Estimation MSE*

To assess the effect of PDR on the channel estimation accuracy, and hence system performance, one has to consider several possible cases of the MMSE estimator design. We begin with derivation of the general MSE formula and then consider three special design scenarios.

As the optimal system is assumed to have equispaced pilots (see Subsection A), MSE expression is the same for all subcarriers in the OFDM spectrum and found [11] to be equal to

$$\begin{aligned}\sigma_{\mathrm{v}}^2 &= \mathrm{E}\left[V_{\mathrm{d}_l} V_{\mathrm{d}_l}^*\right] = \frac{1}{N}\mathrm{E}\left[\left(\hat{\mathbf{H}}-\mathbf{H}\right)^{\mathrm{H}}\left(\hat{\mathbf{H}}-\mathbf{H}\right)\right] = \rho\,\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}^2\right] \\ &- 2\rho^2\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}^2\left(\tilde{\mathbf{R}}_{\mathbf{hh}}^{-1}+\rho\mathbf{I}\right)^{-1}\right] + \rho^3\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}^2\left(\tilde{\mathbf{R}}_{\mathbf{hh}}^{-1}+\rho\mathbf{I}\right)^{-2}\right] \\ &- 2\rho\,\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}\mathbf{R}_{\mathbf{hh}}\right] + 2\rho^2\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}\left(\tilde{\mathbf{R}}_{\mathbf{hh}}^{-1}+\rho\mathbf{I}\right)^{-1}\mathbf{R}_{\mathbf{hh}}\right] \\ &+ \rho^2\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}^2\mathbf{R}_{\mathbf{hh}}\right] - 2\rho^3\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}^2\left(\tilde{\mathbf{R}}_{\mathbf{hh}}^{-1}+\rho\mathbf{I}\right)^{-1}\mathbf{R}_{\mathbf{hh}}\right] \\ &+ \rho^4\mathrm{trace}\left[\tilde{\mathbf{R}}_{\mathbf{hh}}^2\left(\tilde{\mathbf{R}}_{\mathbf{hh}}^{-1}+\rho\mathbf{I}\right)^{-2}\mathbf{R}_{\mathbf{hh}}\right] + \mathrm{trace}\left[\mathbf{R}_{\mathbf{hh}}\right], \quad (12)\end{aligned}$$

where $\rho = SNR_{\mathrm{p}}P$, $\mathbf{R}_{\mathbf{hh}}$ is the true CIR correlation matrix, and $\tilde{\mathbf{R}}_{\mathbf{hh}}$ denotes the CIR correlation matrix used for the estimator design in (8). For the purpose of derivations $\tilde{\mathbf{R}}_{\mathbf{hh}}$ was assumed to be invertible, i.e. full rank. SNR_{p} in (12) is assumed to be known.

The CIR correlation matrices, $\mathbf{R}_{\mathbf{hh}}$ and $\tilde{\mathbf{R}}_{\mathbf{hh}}$, in (12) and the CFR correlation matrices, $\mathbf{R}_{\mathbf{HH}}$ and $\tilde{\mathbf{R}}_{\mathbf{HH}}$, in (8) are related as

$$\mathbf{R}_{\mathbf{HH}} = \mathbf{FBR}_{\mathbf{hh}}\mathbf{B}^{\mathrm{H}}\mathbf{F}^{\mathrm{H}}, \quad (13)$$

$$\tilde{\mathbf{R}}_{\mathbf{HH}} = \mathbf{FB}\tilde{\mathbf{R}}_{\mathbf{hh}}\mathbf{B}^{\mathrm{H}}\mathbf{F}^{\mathrm{H}}, \quad (14)$$

where $\mathbf{F}$ is the $N \times N$-size Fourier matrix with elements $F_{m,n} = \exp(-\mathrm{j}2\pi mn/N)$ and $\mathbf{B} = \left[\begin{matrix}\mathbf{I}_{L\times L} & \mathbf{0}_{L\times(N-L)}\end{matrix}\right]^{\mathrm{T}}$ is the $N \times L$-size padding matrix.

Note that the formula (12) represents a complicated dependence of the MSE on SNR_{p}, which is in turn a monotonically increasing function of PDR (9). To understand the impact of PDR selection on the MSE performance, we need to consider the three practical approaches towards the MMSE estimator design, namely $\tilde{\mathbf{R}}_{\mathbf{hh}}$ choice.

1) Ideal (Matched) Estimator Design

We first derive the MSE for the optimal design case when $\tilde{\mathbf{R}}_{\mathbf{hh}} = \mathbf{R}_{\mathbf{hh}}$, i.e. the CIR correlation is either known or accurately estimated at the OFDM receiver. In this particular scenario the estimator efficiently achieves MMSE at its output given by the following reduction of (12):

$$MMSE = \sum_{m=0}^{L-1}\frac{1}{\lambda_m^{-1}+\rho} = \sum_{m=0}^{L-1}\frac{\lambda_m}{1+\lambda_m\rho}, \quad (15)$$

where λ_m, $m \in [0; L-1]$ are the eigenvalues of $\mathbf{R}_{\mathbf{hh}}$. Note that in contrast to (12) the latter expression form in (15) allows for $\lambda_m = 0$, i.e. the case when $\mathbf{R}_{\mathbf{hh}}$ is not of the full rank. Channels characterised by such a correlation matrix are known as sparse due to the presence of only limited number (< L) of the independent multipath components.

It is important to note that the MMSE function (15) with respect to the λ_m value distribution and subject to the energy-normalising condition, $\sum_{m=0}^{L-1}\lambda_m = 1$, reaches a unique maximum of $MMSE_{\max} = 1/(1+\rho/L)$ at $\lambda_m = 1/L$ for all $m \in [0; L-1]$. It could be interpreted as if there were L independent identically distributed multipath components. If, however, one or several eigenvalues are close to zero, then MMSE tends to decrease towards the smallest value of $MMSE_{\min} = 1/(1+\rho)$ corresponding to only one path underlying the CIR (flat fading channel). Note also that at the higher operational SNR_{p} values the MMSE difference between various λ_m distributions becomes smaller due to domination of the ρ factor in (15).

2) Half-matched Estimator Design

An optimal MMSE estimator design is complex due to the necessity to invert a CIR or CFR correlation matrix that has to be accurately estimated itself. In one of our previous works [14] a simplified suboptimal scheme has been proposed for the sample-spaced multipath channels that treats the CIR samples as if they were independent random Gaussian variables. Hence $\tilde{\mathbf{R}}_{\mathbf{hh}}$ is constructed to be diagonal and therefore easily invertible. $\tilde{\mathbf{R}}_{\mathbf{hh}}$ entries represent the measured power-delay profile (PDP) samples. For the simplicity of analysis we assume ideally determined PDP, i.e. $[\tilde{\mathbf{R}}_{\mathbf{hh}}]_{k,l} = \begin{cases} [\mathbf{R}_{\mathbf{hh}}]_{k,l} = r_k & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}$, $k, l \in [0; L-1]$. Then (12) is reduced to

$$MSE_{\text{HM}} = \sum_{m=0}^{L-1} \frac{1}{r_m^{-1} + \rho} = \sum_{m=0}^{L-1} \frac{r_m}{1 + r_m \rho} \,. \tag{16}$$

In analogy with the MMSE function (15), taking into account CIR energy normalisation $\sum_{m=0}^{L-1} r_m = 1$, one can find the upper and lower bounds as $MSE_{\text{HM max}} = 1/(1+\rho/L)$ and $MSE_{\text{HM min}} = 1/(1+\rho)$.

In most practical scenarios of the generally non-sample-spaced channels (both dense and sparse) $r_m > 0$ for all $m \in [0; L-1]$, due to the sinc-interpolation of the multipath components (3) yielding the bandlimited CIR with mutually correlated samples. Thus, for a random propagation environment MSE_{HM} (16) would take a value closer to the upper bound, $MSE_{\text{HM max}} = MMSE_{\text{max}}$, with much higher probability than $MMSE$ (15) corresponding to the optimal estimator design. In a special case of the sample-spaced multipaths (e.g., peculiar to the ultra-wideband systems) the diagonal-form estimator (16) becomes optimal, i.e. $MSE_{\text{HM}} = MMSE$, greatly improving performance over the sparse channels.

3) Mismatched Estimator Design

Finally let us consider the estimator design that does not require any correlation knowledge. In a number of research works [5], [6] it has been reported that the uniform PDP design assuming all CIR components to be independently identically distributed yields MSE performance equal to the upper bound $MMSE_{\text{max}}$ for any kind of the channel response and is called robust. We confirm this result by setting $\tilde{\mathbf{R}}_{\mathbf{hh}} = L^{-1}\mathbf{I}$ and calculating estimation MSE for an arbitrary CIR correlation described by $\mathbf{R}_{\mathbf{hh}}$ in (12):

$$MSE_{\text{M}} = \frac{1}{1+\rho/L} - \frac{1 - \sum_{m=0}^{L-1} \lambda_m}{(1+\rho/L)^2} \,. \tag{17}$$

If the normalisation condition, $\sum_{m=0}^{L-1} \lambda_m = 1$, is satisfied, then $MSE_{\text{M}} = MMSE_{\text{max}} = 1/(1+\rho/L)$, i.e. estimation MSE is indeed independent of the power distribution within the actual CIR. It is interesting to note that $MSE_{\text{M}} \le MMSE_{\text{max}}$ only if the average CIR energy does not exceed unity given the design matrix $\tilde{\mathbf{R}}_{\mathbf{hh}} = L^{-1}\mathbf{I}$. This circumstance is important to account for in case of the imperfect received signal power gain at the input of the channel estimator. Under-amplification is recommended rather than the over-amplification.

Summarising MSE performance for all the three design cases one has to note the upper MSE bound ($MMSE_{\text{max}}$) as a monotonically decreasing function of PDR to be a quantity of interest for the optimal PDR search. Indeed $MMSE_{\text{max}}$ reflects the variance of the channel estimation error both for the optimal estimator design when all CIR samples are mutually independent (the most performance-challenging scenario) and for the design accommodating arbitrary CIR correlation (the robust operation mode).

C. Equaliser SNR Maximisation

The instantaneous SNR at the output of the equaliser is expressed from (6) as

$$\gamma_{\text{d}_l}(i) = \left|\frac{X_{\text{d}_l}(i)}{\Psi_{\text{d}_l}(i)}\right|^2 = \frac{\left|\hat{H}_{\text{d}_l}(i)\right|^2 \left|X_{\text{d}_l}(i)\right|^2}{\left|X_{\text{d}_l}(i)\, V_{\text{d}_l}(i) + W_{\text{d}_l}(i)\right|^2} \,, \tag{18}$$

where $X_{\text{d}_l}(i)$, $V_{\text{d}_l}(i)$ and $W_{\text{d}_l}(i)$, as well as $X_{\text{d}_l}(i)$ and $\hat{H}_{\text{d}_l}(i)$ are mutually independent random variables as the channel estimation algorithm uses a subset of the pilot subcarriers that is distinct from the data subcarriers.

An obvious criterion to maximise could be the average SNR at the equaliser output $\gamma = \text{E}\left[\left|X_{\text{d}_l}(i)\right|^2\right] \Big/ \text{E}\left[\left|\Psi_{\text{d}_l}(i)\right|^2\right]$. However, due to the mathematical challenges incurred by calculating $\text{E}\left[\left|\Psi_{\text{d}_l}(i)\right|^2\right]$ we use an equivalent suboptimal criterion instead defined in the form of

$$\begin{aligned} \bar{\gamma} &= \frac{\text{E}\left[\left|\hat{H}_{\text{d}_l}(i)\right|^2 \left|X_{\text{d}_l}(i)\right|^2\right]}{\text{E}\left[\left|X_{\text{d}_l}(i)\, V_{\text{d}_l}(i) + W_{\text{d}_l}(i)\right|^2\right]} \\ &= \frac{\text{E}\left[\left|H_{\text{d}_l}(i) - V_{\text{d}_l}(i)\right|^2\right] \text{E}\left[\left|X_{\text{d}_l}(i)\right|^2\right] \Big/ \text{E}\left[\left|W_{\text{d}_l}(i)\right|^2\right]}{1 + \text{E}\left[\left|V_{\text{d}_l}(i)\right|^2\right] \text{E}\left[\left|X_{\text{d}_l}(i)\right|^2\right] \Big/ \text{E}\left[\left|W_{\text{d}_l}(i)\right|^2\right]} \\ &= \frac{SNR_{\text{d}}\left(1 - \sigma_{\text{v}}^2\right)}{1 + SNR_{\text{d}}\sigma_{\text{v}}^2} \,, \end{aligned} \tag{19}$$

where $\text{E}\left[\left|H_{\text{d}_l}(i) - V_{\text{d}_l}(i)\right|^2\right] = \text{E}\left[\left|H_{d_l}(i)\right|^2\right] - \text{E}\left[\left|V_{d_l}(i)\right|^2\right]$ in the numerator follows from the definition of the MMSE algorithm, according to which the estimation error is orthogonal to the estimate. $SNR_{\text{d}} = \text{E}\left[\left|X_{\text{d}_l}(i)\right|^2\right] \Big/ \text{E}\left[\left|W_{\text{d}_l}(i)\right|^2\right] = \bar{\sigma}_{\text{d}}^2 / \sigma_{\text{w}}^2$ in (19) represents the average SNR observed at the data subcarriers and is related to the SNR observed at the receiver input according to the expression:

$$SNR_{\mathrm{d}} = SNR_{\mathrm{p}} \big/ \kappa = \frac{N}{P\kappa + N - P} SNR \,. \tag{20}$$

Analysis of the $\bar{\gamma}$ function (19) reveals the existence of a global maximum with respect to PDR explained as follows. SNR_{d} and σ_{v}^2 have been previously defined as monotonically decreasing functions of PDR. Therefore $\bar{\gamma}$, as an increasing function of SNR_{d}, decreases with PDR enlargement, whereas exhibits growth as a descending function of σ_{v}^2. Obviously it is of practical interest to ensure an optimal PDR selection for the worst-case scenario, i.e. when σ_{v}^2 takes the largest value of $MMSE_{\max}$ yielding minimum of $\bar{\gamma}$. Thus, one can meet the requirements of the robust operational mode for the receiver. Furthermore, as will be seen from the numerical examples, although for the sparse channels and optimal MMSE estimator design optimisation of $\bar{\gamma}$ letting $\sigma_{\mathrm{v}}^2 = MMSE_{\max}$ results in an over-determined PDR solution, the SNR at the equaliser's output is still kept close to the maximum due to the near-flat functional region around the optimal PDR point.

We derive the optimal PDR by searching for a maximum of $\bar{\gamma}$ through the differentiation procedure with respect to κ and solving the resultant quadratic equation:

$$\kappa_{\mathrm{opt}} = \arg\max_{\kappa > 0} \bar{\gamma} \,, \tag{21}$$

where $\bar{\gamma} = \dfrac{SNR_{\mathrm{d}}\left(1 - MMSE_{\max}\right)}{1 + SNR_{\mathrm{d}} MMSE_{\max}}$.

Substituting $MMSE_{\max} = L/(L+\rho)$ and (20) in (21), we obtain

$$\bar{\gamma} = \frac{PN^2 SNR^2 \kappa}{\left[P\kappa + N - P\right]\left[P(L + N\,SNR)\kappa + LN(1 + SNR) - LP\right]} \,. \tag{22}$$

Subsequently solving $\partial\bar{\gamma}/\partial\kappa = 0$ results in

$$\kappa_{\mathrm{opt}} = \frac{(N-P)}{P}\sqrt{\frac{1 + \dfrac{N}{N-P} SNR}{1 + \dfrac{N}{L} SNR}} \,. \tag{23}$$

Depending on the SNR magnitude ($0 < SNR < \infty$) optimal PDR (23) takes a value in the interval $(N-P)/P < \kappa_{\mathrm{opt}} < \sqrt{L(N-P)}\big/P$.

V. Numerical Results

The PSAM-based OFDM system under practical consideration has 64 subcarriers. 16 of them are allocated for the pilot signals. Subcarriers transmitting uncoded data are modulated by the Gray-mapped 16QAM. CP length is chosen to be equal to 16 samples.

The channel is modelled as quasi-static with no correlation between the neighbouring OFDM symbol transmissions. PDP of the multipath response vector $\mathbf{a}(i)$ is selected to be highly sparse (with only two non-zero paths) and uniform, i.e. $\mathrm{E}\left[\mathbf{a}(i)\mathbf{a}(i)^{\mathrm{H}}\right] = 1/2\,\mathbf{I}_{2\times 2}$. The rms delay spread of the CIR is equal to $\tau_{\mathrm{rms}} = 3.2T/N$ that corresponds to the 6.4-sample long delay between the multipath components. The CIR energy is strictly constrained within the CP length (16 samples) and any outer leakage due to the bandlimiting is truncated.

The operational E_{b}/N_0 range is varied from 5 to 30dB. The design of the channel estimator is made assuming SNR at the receiver input to be known. We also consider both optimal MMSE estimator design ($\tilde{\mathbf{R}}_{\mathbf{hh}} = \mathbf{R}_{\mathbf{hh}}$) and robust design ($\tilde{\mathbf{R}}_{\mathbf{hh}} = 1/16\,\mathbf{I}_{16\times 16}$).

Fig. 1 illustrates the theoretical dependence of the optimal PDR (23) on the number of pilot subcarriers. Here we consider only the optimal system design case when it is anticipated that $P = L$ (see Section IVA).

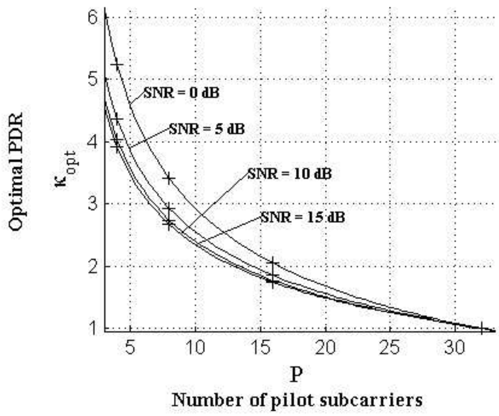

Fig. 1. Theoretical PDR dependence on the number of pilot subcarriers (equispaced pilot modes are denoted by +'s)

It follows from Fig. 1 that increase in the pilot number allows for PDR lowering. For the weakly dispersive channels (described by small L) a few boosted pilot subcarriers are sufficient. It is important to note that the optimal PDR setting is almost invariant in respect to SNR at the receiver input for most of the operational SNR range. Furthermore the dependence on SNR disappears for more dispersive channels.

To assess conformity of the criterion (19) to yield an optimal PDR, we measure actual SNR (γ) at the equaliser output in the simulated system. The results are shown in Fig. 2 in the equaliser-output-to-receiver-input normalised form. In case of the robust channel estimator design SNR maximum region surrounds the optimal PDR calculated analytically (23). It has also been found that the operational SNR has no substantial influence on the optimal PDR choice leading to only a minor shift within the flat performance region. Hence lower operational SNR bound has to be taken into account as it guarantees PDR to lie in the closest-to-maximum performance region for higher SNRs.

Knowledge of the CIR correlation underlying optimal MMSE estimator design could potentially yield a more precise setting for the pilot power. As can be seen from Fig. 2 MSE reduction in case of the optimally estimated sparse CIRs allows for the data subcarrier power to be increased with respect to the pilot power and hence improves detection performance by up to 2.5dB in comparison with the robust mode. However this observation is true only for the selected channel model. For other channels with less sparsity optimal PDR will be closer or equal to the optimal

PDR calculated for the robust mode.

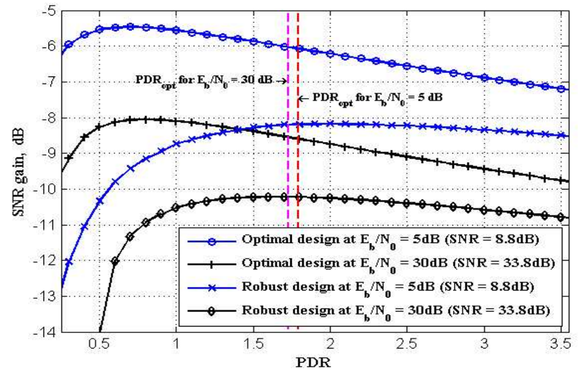

Fig. 2. SNR gain of the equaliser output in respect to the receiver input

Finally, BER results for a variety of PDR settings are presented in Fig. 3. The following PDRs have been tested:

- equal power allocation between pilot and data subsymbols (PDR = 1);
- optimum PDR determined for $E_b/N_0 = 5\text{dB}$ (PDR = 1.79);
- pilot power equal to the minimum 16QAM power (PDR = 0.2);
- pilot power equal to the doubled maximum 16QAM power (PDR = 3.6).

Note that specifically in our case the optimum pilot power is nearly equal to the maximum 16QAM power $\sigma^2_{\max}$, i.e. $\kappa_{\text{opt}} \approx PAR$ (10). In general, however, it is not true (Fig. 1).

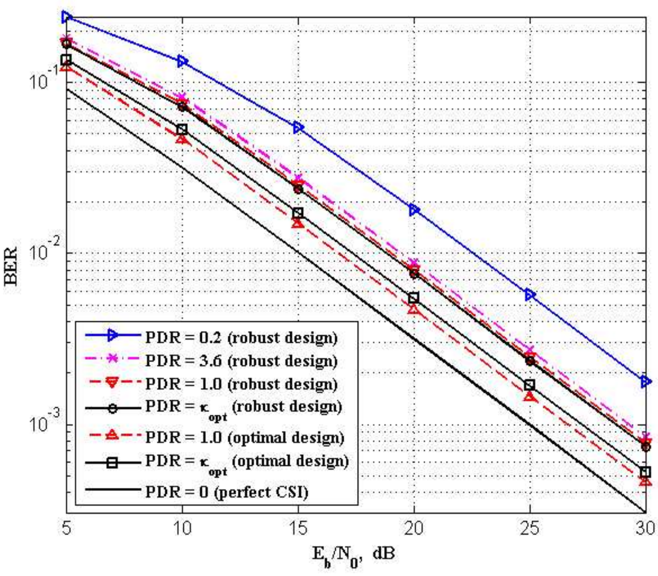

Fig. 3. BER performance for various PDR settings

Simulation results (Fig. 3) confirm that the optimal PDR maximises BER performance in case of the robust estimator design irrespective of SNR at the receiver input. Nevertheless one can see that considerable overdetermining of PDR still results in performance close to the optimal PDR choice. In contrast to the robust mode, for the optimal MMSE estimator design performance is found to be closer to the case with perfect channel state information (CSI) at the receiver. Here equal power allocation leads to slightly better BER than PDR calculated according to (23) as PDR = 1 is closer to the actual optimum. However the difference is quite small and for other channel responses, as well as for the inaccurately estimated noise variance an optimal value approaches (23).

VI. Conclusion

In this paper, the optimum pilot-to-data power ratio has been derived for the OFDM systems with robust MMSE channel estimation. It has been found that optimal PDR setting is almost insensitive to the receiver input SNR and is tolerable to the overdetermining.

Acknowledgment

The authors would like to thank the University of Cape Town, Telkom SA, Siemens, the National Research Foundation (NRF) and the Department of Trade and Industry (DTI) for providing financial support for this research.

References

[1] M. Engels, *Wireless OFDM Systems: How to Make Them Work?* IMEC, Belgium, 2002.

[2] Y. Mostofi, and D. C. Cox, “Average error rate analysis for pilot-aided OFDM receivers with frequency-domain interpolation,” in *Proc. IEEE Wireless Commun. and Netw. Conf. (WCNC)*, pp. 1421-1425, Mar. 2004.

[3] X. Wang, and K. J. R. Liu, “Channel estimation for multicarrier modulation systems using a time-frequency polynomial model,” *IEEE Trans. Commun.*, vol. 50, pp. 1045-1049, Jul. 2002.

[4] M.-X. Chang, and Y. T. Su, “Model-based channel estimation for OFDM signals in Rayleigh fading,” *IEEE Trans. Commun.*, vol. 50, pp. 540-545, Apr. 2002.

[5] O. Edfords, M. Sandell, J.-J. van de Beek, S. K. Wilson, and P. O. Börjesson, “OFDM channel estimation by singular value decomposition,” *IEEE Trans. Commun.*, vol. 46, pp. 931–939, Jul. 1998.

[6] Y. (G.) Li, L. J. Cimini Jr., and N.R. Sollenberger, “Robust channel estimation for OFDM systems with rapid dispersive fading channels,” *IEEE Trans. Commun.*, vol. 46, pp. 902-915, Jul. 1998.

[7] L. Huang, G. Mathew, and J. W. M. Bergmans, “Pilot-aided channel estimation for systems with virtual carriers,” in *Proc. IEEE Intern. Conf. Commun. (ICC)*, pp. 3070-3075, Jun. 2006.

[8] L. Deneire, P. Vandenameele, L. van der Perre, B. Gyselinckx, and M. Engels, “A low complexity ML channel estimator for OFDM communications,” *IEEE Trans. Commun.*, vol. 51, pp. 135-140, Feb. 2003.

[9] P. Chen, and H. Kobayashi, “Maximum likelihood channel estimation and signal detection for OFDM systems,” in *Proc. IEEE Intern. Conf. Commun. (ICC)*, pp. 1640-1645, Apr. 2002.

[10] E. Golovins, and N. Ventura, “Comparative analysis of low complexity channel estimation techniques for the pilot-assisted wireless OFDM systems,” in *Proc. Southern African Telecomm. Networks Applications Conf. (SATNAC)*, Sep. 2006.

[11] E. Golovins, and N. Ventura, “Design and performance analysis of low-complexity pilot-aided OFDM channel estimators,” in *Proc. 6th IEEE Intern. Workshop on Multi-Carrier and Spread Spectrum (MC-SS)*, May 2007.

[12] S. Ohno, and G. B. Giannakis, “Capacity maximizing MMSE-optimal pilots for wireless OFDM over frequency-selective block Rayleigh-fading channels,” *IEEE Trans. Inform. Theory*, vol. 50, pp. 2138-2145, Sep. 2004.

[13] X. Cai, and G. B. Giannakis, “Error probability minimizing pilots for OFDM with M-PSK modulation over Rayleigh-fading channels,” *IEEE Trans. Veh. Technol.*, vol. 53, pp. 146-155, Jan. 2004.

[14] E. Golovins, and N. Ventura, “Low-complexity constrained LMMSE estimator for the sparse OFDM channels,” to appear in *Proc. IEEE Africon 2007 Conf.*, Sep. 2007.

Eugene Golovins received B.S. and M.S. degrees with distinction in electrical engineering from Riga Technical University, Latvia, in 2002 and 2004, respectively. He is currently working towards the Ph.D. degree in electrical engineering at the University of Cape Town, South Africa. He is also a research assistant at the Department of Electrical Engineering, University of Cape Town.

His research interests are in the area of wireless communication systems and networks, as well as fiber-optical communications.